Pricing American Options under the Constant Elasticity of Variance Model and subject to Bankruptcy*

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Abstract

This paper proposes an alternative characterization of the early exercise premium that is valid for any Markovian and diffusion underlying price process as well as for any parameterization of the exercise boundary. This new representation is shown to provide the best pricing alternative available in the literature for medium- and long-term American option contracts, under the Constant Elasticity of Variance model. Moreover, the proposed pricing methodology is also extended easily to the valuation of American options on defaultable equity, and possesses appropriate asymptotic properties.

I. Introduction

This paper proposes a new analytical approximation for the American option value that can be applied under any Markovian and diffusion underlying asset price process. In contrast with the previous literature, the proposed characterization of the American option can accommodate the risk of default attached to the underlying equity, and is shown to converge to the exact perpetual solution, being therefore extremely accurate, even for long-term contracts.

The absence of an exact and closed-form pricing solution for the American put (or call, but on a dividend-paying asset) stems from the fact that the option price and the early exercise boundary must be determined simultaneously as the solution of the same free boundary problem set up by McKean (1965). Consequently, the vast literature on this subject, which is reviewed for instance in Barone-Adesi (2005), has proposed only numerical solution methods and analytical approximations.

The numerical methods include the finite difference schemes introduced by Brennan and Schwartz (1977), and the binomial model of Cox, Ross, and Rubinstein (1979). These methods are both simple and convergent, but they are also too time-consuming and do not provide the comparative statics attached to an analytical solution.

One of the first analytical approximations is due to Barone-Adesi and Whaley (1987), who use the quadratic method of MacMillan (1986). Despite its high efficiency and the accuracy improvements brought by subsequent extensions (see for example, Ju and Zhong (1999)), this method is not convergent. Johnson (1983) and Broadie and Detemple (1996) provide lower and upper bounds for American options, which are based on regression coefficients that are estimated through a time-demanding calibration to a large set of options contracts. As argued in Ju ((1998), p. 642), this econometric approach is not convergent and can generate less accurate hedging ratios, because the regression coefficients are optimized only for pricing purposes. More recently, Sullivan (2000) approximates the option value function through Chebyshev polynomials and employs a Gaussian quadrature integration scheme at each discrete exercise date. Although the speed and accuracy of the proposed numerical approximation can be enhanced via Richardson extrapolation, its convergence properties are still unknown.

Geske and Johnson (1984) approximate the American option price through an infinite series of multivariate normal distribution functions. Although convergence can be insured by adding more terms, only the first few terms are considered, and a Richardson extrapolation scheme is employed in order to reduce the computational burden.[1] Another fast and accurate convergent method is the randomization approach of Carr (1998), which also uses Richardson extrapolation. It must be noted, however, that one of the main disadvantages of extrapolation schemes is the indetermination of the sign for the approximation error.

Kim (1990), Jacka (1991), Carr, Jarrow, and Myneni (1992), and Jamshidian (1992) proposed the so-called "integral representation method" to describe the early exercise premium. However, the numerical efficiency of this approach depends on the specification adopted for the early exercise boundary. For instance, Ju (1998) derives fast and accurate approximate solutions based on a multipiece exponential representation of the early exercise boundary.

All the studies mentioned are based on the Black and Scholes (1973) model, and most of them differ only in the specification adopted for the exercise boundary. Kim and Yu (1996) and Detemple and Tian (2002) constitute two notable exceptions: they extend the "integral representation method" to alternative diffusion processes. However, and in opposition to the standard geometric Brownian motion case, such an extension does not offer a closed-form solution for the integral equation characterizing the early exercise premium, which undermines the computational efficiency of this approach.

Based on the optimal stopping approach initiated by Bensoussan (1984) and Karatzas (1988), this paper derives an alternative characterization of the American option price that is valid for any continuous representation of the exercise boundary and for any Markovian (and diffusion) price process describing the dynamics of the underlying asset price. The proposed characterization possesses at least three advantages over the extended integral representation

[1] Chung and Shackleton (2007) generalize the Geske-Johnson method through a two-point scheme based not only on the inter-exercise time dimension, but also on the time to maturity of the option contract.

of Kim and Yu ((1996), equations 10 or 13): 1) it converges to the perpetual American option price as the option maturity tends to infinity; 2) its accuracy does not deteriorate as the option maturity is lengthened; and 3) it can be adapted easily to the context of defaultable stock options pricing models. Although knowledge of the first passage time density of the underlying price process to the exercise boundary is required by the proposed pricing solution, it is shown that such optimal stopping time density can be recovered easily from the transition density function. Hence, the proposed characterization of the American option price requires only an efficient valuation formula for its European counterpart, as well as knowledge of the underlying asset price transition density function.

To exemplify the proposed pricing methodology, several parameterizations of the early exercise boundary are tested under the usual geometric Brownian motion assumption and the Constant Elasticity of Variance (CEV) model. Special attention is devoted to this latter framework since it is consistent with two well known facts that have found empirical support in the literature: the existence of a negative correlation between stock returns and realized stock volatility (*leverage effect*), as documented, for instance, in Bekaert and Wu (2000); and the inverse relation between the implied volatility and the strike price of an option contract (*implied volatility skew*), which is observed, for example, by Dennis and Mayhew (2002).

Although the pricing of European options under the CEV process has become well established since the seminal work of Cox (1975), the same cannot be said about the valuation of American options. As noted by Nelson and Ramaswamy ((1990), p. 418), the *simple binomial processes* approximation proposed by these authors becomes inaccurate as option maturity is increased. Alternatively, and as shown in Section VI, the integral approach suggested by Kim and Yu ((1996), subsection 3.4) and Detemple and Tian ((2002), Proposition 3) is less efficient than the proposed pricing methodology, unless such recursive scheme is accelerated through Richardson extrapolation, in which case its accuracy may deteriorate for medium- and long-term options. This paper is intended to fill this gap in the literature. Additionally, the optimal stopping approach presented in this paper is also adapted to the context of the *jump to default extended CEV* model (JDCEV) proposed by Carr and Linetsky (2006). This extension of the literature provides analytical pricing solutions for

American options on defaultable equity, which are consistent with both the aforementioned leverage effect, and the positive relationship between default indicators and equity volatility that is documented, for instance, by Campbell and Taksler (2003).

This paper proceeds as follows. Based on the optimal stopping approach, Section II separates the American option into a non-deferrable rebate and a European down-and-out option. In Section III, such representation is shown to be equivalent to the usual decomposition between a European option and an early exercise premium. A new analytical characterization is offered for the early exercise premium, and its asymptotic properties are tested. Section IV provides an efficient algorithm to recover the first hitting time density of the underlying price process, which allows the comparison, in Section VI, of the different specifications of the early exercise boundary discussed in Section V. Section VII extends the new representation of the early exercise premium to the JDCEV model, and Section VIII concludes.

II. Model Setup

The valuation of American options will be first explored in the context of a stochastic intertemporal economy with continuous trading on the time-interval $[t_0, T]$, for some fixed time $T > t_0$, where uncertainty is represented by a complete probability space $(\Omega, \mathcal{F}, \mathbb{Q})$. Throughout the paper, $\mathbb{Q}$ will denote the martingale probability measure obtained when the numéraire of the economy under analysis is taken to be a money market account B_t, whose dynamics are governed by the following ordinary differential equation:

$$dB_t = rB_t dt, \tag{1}$$

where r denotes the riskless interest rate, which is assumed to be constant.

Although the alternative representation of the early exercise premium that will be proposed in Proposition 1 requires only that the underlying asset price process S_t be a Markovian diffusion, the subsequent empirical analysis will be based on the following one-dimensional diffusion process:

$$\frac{dS_t}{S_t} = (r - q)\, dt + \sigma(t, S)\, dW_t^{\mathbb{Q}}, \tag{2}$$

where q represents the dividend yield for the asset price, $\sigma(t, S)$ corresponds to the instantaneous volatility (per unit of time) of asset returns and $W_t^{\mathbb{Q}} \in \mathbb{R}$ is a standard Brownian motion, initialized at zero and generating the augmented, right continuous, and complete filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq t_0\}$. Nevertheless, equation (2) encompasses several well known option pricing models as special cases: for example, it corresponds to the geometric Brownian motion if $\sigma(t, S) = \sigma$ is a constant; and it yields the CEV process when

$$\sigma(t, S) = \delta S_t^{\frac{\beta}{2}-1}, \tag{3}$$

for $\delta, \beta \in \mathbb{R}$.[2]

Hereafter, the analysis will focus on the valuation of an American option on the asset price S, with strike price K, and with maturity date T, whose time-t $(\leq T)$ value will be denoted by $V_t(S, K, T; \phi)$, where $\phi = -1$ for an American call or $\phi = 1$ for an American put. Since the American option can be exercised at any time during its life, it is well known—see, for example, Karatzas ((1988), Theorem 5.4)—that its price can be represented by the *Snell envelope*:

$$V_{t_0}(S, K, T; \phi) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}} \left\{ e^{-r[(T \wedge \tau) - t_0]} (\phi K - \phi S_{T \wedge \tau})^+ \middle| \mathcal{F}_{t_0} \right\}, \tag{4}$$

where $\mathcal{T}$ is the set of all stopping times for the filtration $\mathbb{F}$ generated by the underlying price process and taking values in $[t_0, \infty]$.[3]

Since the underlying asset price is a diffusion and both interest rates and dividend yields are assumed to be deterministic, for each time $t \in [t_0, T]$ there exists a *critical asset price* E_t below (above) which the American put (call) price equals its intrinsic value and, therefore, early exercise should occur—see, for instance, Carr, Jarrow, and Myneni ((1992), equations 1.2 and 1.3). Consequently, the optimal policy should be to exercise the American option

[2] The underlying asset can be thought of as a stock, a stock index, an exchange rate, or a financial futures contract, so long as the parameter q is understood as, respectively, a dividend yield, an average dividend yield, the foreign default-free interest rate, or the domestic risk-free interest rate.

[3] $\mathbb{E}_{\mathbb{Q}}(X | \mathcal{F}_t)$ denotes the expected value of the random variable X, conditional on $\mathcal{F}_t$, and computed under the equivalent martingale measure $\mathbb{Q}$. Similarly, $\mathbb{Q}(\omega | \mathcal{F}_t)$ will represent the probability of event ω, conditional on $\mathcal{F}_t$, and computed under the probability measure $\mathbb{Q}$.

when the underlying asset price first touches its critical level. Representing the first passage time of the underlying asset price to its moving boundary by

$$\tau_e := \inf\{t > t_0 : S_t = E_t\} \tag{5}$$

and considering that the American option is still alive at the valuation date (i.e., $\phi S_{t_0} > \phi E_{t_0}$), equation (4) can then be restated as:

$$\begin{aligned} V_{t_0}(S, K, T; \phi) &= \mathbb{E}_{\mathbb{Q}}\left\{ e^{-r[(T\wedge\tau_e)-t_0]} (\phi K - \phi S_{T\wedge\tau_e})^+ \middle| \mathcal{F}_{t_0} \right\} \\ &= \mathbb{E}_{\mathbb{Q}}\left[e^{-r(\tau_e - t_0)} (\phi K - \phi E_{\tau_e}) \mathbb{1}_{\{\tau_e < T\}} \middle| \mathcal{F}_{t_0} \right] \\ &\quad + e^{-r(T-t_0)} \mathbb{E}_{\mathbb{Q}}\left[(\phi K - \phi S_T)^+ \mathbb{1}_{\{\tau_e \geq T\}} \middle| \mathcal{F}_{t_0} \right], \end{aligned} \tag{6}$$

where the first line of equation (6) follows from equation (5), and $\mathbb{1}_{\{A\}}$ denotes the indicator function of the set A. Note that $K \geq E_{\tau_e}$ for the American put, because the exercise boundary is limited from above by $\min\left(K, \frac{r}{q}K\right)$—see, for instance, Huang, Subrahmanyam, and Yu ((1996), footnote 5). For the American call, $K \leq E_{\tau_e}$ because the early exercise boundary is limited from below by $\max\left(K, \frac{r}{q}K\right)$—see, for example, Kim and Yu ((1996), p. 67).

For $\phi = 1$, equation (6) is equivalent to Kim and Yu ((1996), eq. 7) and decomposes the American put into two components. The first one corresponds to the present value of a non-deferrable (and, in general, also non-constant) rebate $(K - E_{\tau_e})$, payable at the optimal stopping time τ_e. The second component is simply the time-t_0 price of a European down-and-out put on the asset S, with strike price K, maturity date at time T, and (time-dependent) barrier levels $\{E_t, t_0 \leq t \leq T\}$. Assuming a convenient parametric specification for the barrier function E_t, it is possible to convert equation (6) into a closed-form solution. Such an approach was pursued, for instance, by Ingersoll (1998) using both constant and exponential specifications, and by Sbuelz (2004), also under a constant barrier formulation. Unfortunately, the time path $\{E_t, t_0 \leq t \leq T\}$ of critical asset prices, which is called the *exercise boundary,* is not known *ex ante* and therefore the assumption of a specific parametric form for the barrier function simply transforms equation (6) into a lower bound for the true American put option value.

III. The Early Exercise Premium

Similarly to Kim (1990), Jacka (1991), and Carr, Jarrow, and Myneni (1992), the American option price can be divided into two components: the corresponding European option price and an early exercise premium. For this purpose, and because

$$\mathbb{1}_{\{\tau_e \geq T\}} = 1 - \mathbb{1}_{\{\tau_e < T\}}, \tag{7}$$

equation (6) can be rewritten as:

$$\begin{aligned} V_{t_0}(S, K, T; \phi) &= \mathbb{E}_{\mathbb{Q}}\left[e^{-r(\tau_e - t_0)} (\phi K - \phi E_{\tau_e}) \mathbb{1}_{\{\tau_e < T\}} \middle| \mathcal{F}_{t_0} \right] \\ &\quad + e^{-r(T-t_0)} \mathbb{E}_{\mathbb{Q}}\left[(\phi K - \phi S_T)^+ \middle| \mathcal{F}_{t_0} \right] \\ &\quad - e^{-r(T-t_0)} \mathbb{E}_{\mathbb{Q}}\left[(\phi K - \phi S_T)^+ \mathbb{1}_{\{\tau_e < T\}} \middle| \mathcal{F}_{t_0} \right]. \end{aligned}$$

And, since

$$e^{-r(T-t_0)} \mathbb{E}_{\mathbb{Q}}\left[(\phi K - \phi S_T)^+ \middle| \mathcal{F}_{t_0} \right] := v_{t_0}(S, K, T; \phi) \tag{8}$$

can be understood (under a deterministic interest rate setting) as the time-t_0 price of the corresponding European option (with technical features identical to those of the American contract under analysis), then

$$\begin{aligned} V_{t_0}(S, K, T; \phi) &= v_{t_0}(S, K, T; \phi) \\ &\quad + \mathbb{E}_{\mathbb{Q}}\left[e^{-r(\tau_e - t_0)} (\phi K - \phi E_{\tau_e}) \mathbb{1}_{\{\tau_e < T\}} \middle| \mathcal{F}_{t_0} \right] \\ &\quad - e^{-r(T-t_0)} \mathbb{E}_{\mathbb{Q}}\left[(\phi K - \phi S_T)^+ \mathbb{1}_{\{\tau_e < T\}} \middle| \mathcal{F}_{t_0} \right]. \end{aligned} \tag{9}$$

The last two terms on the right-hand side of equation (9) correspond to the early exercise premium, for which an analytical solution will be proposed in the next proposition.

A. An Alternative Characterization

The proposition presented below provides a new characterization for the early exercise premium.

Proposition 1 *Assuming that the underlying asset price S_t follows a Markovian diffusion process and that the interest rate r is constant, the time-t_0 value of an American option $V_{t_0}(S,K,T;\phi)$ on the asset price S, with strike price K, and with maturity date T can be decomposed into the corresponding European option price $v_{t_0}(S,K,T;\phi)$ and the early exercise premium $EEP_{t_0}(S,K,T;\phi)$, i.e.,*

$$V_{t_0}(S,K,T;\phi) = v_{t_0}(S,K,T;\phi) + EEP_{t_0}(S,K,T;\phi), \tag{10}$$

with

$$EEP_{t_0}(S,K,T;\phi) := \int_{t_0}^{T} e^{-r(u-t_0)} \left[(\phi K - \phi E_u) - v_u(E,K,T;\phi)\right] \mathbb{Q}\left(\tau_e \in du \middle| \mathcal{F}_{t_0}\right), \tag{11}$$

where $\mathbb{Q}(\tau_e \in du | \mathcal{F}_{t_0})$ represents the probability density function of the first passage time τ_e, as defined by equation (5), $\phi = -1$ for an American call and $\phi = 1$ for an American put.

Proof. Noting that the only random variable contained in the second term on the right-hand side of equation (9) is the first passage time, then

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-r(\tau_e-t_0)}(\phi K - \phi E_{\tau_e})\mathbb{1}_{\{\tau_e<T\}} \middle| \mathcal{F}_{t_0}\right] = \int_{t_0}^{T} e^{-r(u-t_0)}(\phi K - \phi E_u)\,\mathbb{Q}\left(\tau_e \in du \middle| \mathcal{F}_{t_0}\right). \tag{12}$$

Concerning the third term on the right-hand side of equation (9), it is necessary to consider the joint density of the two random variables involved: the first passage time τ_e and the terminal asset price S_T. Hence,

$$\mathbb{E}_{\mathbb{Q}}\left[(\phi K - \phi S_T)^+ \mathbb{1}_{\{\tau_e<T\}} \middle| \mathcal{F}_{t_0}\right] = \int_{\mathbb{R}} (\phi K - \phi S)^+ \,\mathbb{Q}\left(S_T \in dS, \tau_e < T \middle| \mathcal{F}_{t_0}\right), \tag{13}$$

where the integration can be restricted to the domain $\mathbb{R}_+$ if, for example, the geometric Brownian motion assumption is imposed. Because the underlying asset price is assumed to be a Markov process, the joint density contained in equation (13) is simply the convolution between the density of the first passage time τ_e and the transition probability density function of the terminal asset price S_T:

$$\mathbb{Q}\left(S_T \in dS, \tau_e < T \middle| \mathcal{F}_{t_0}\right) = \int_{t_0}^{T} \mathbb{Q}\left(S_T \in dS \middle| S_u = E_u\right)\mathbb{Q}\left(\tau_e \in du \middle| \mathcal{F}_{t_0}\right). \tag{14}$$

Therefore, combining equations (13) and (14),

$$
\begin{aligned}
& \mathbb{E}_{\mathbb{Q}}\left[(\phi K-\phi S_T)^{+}\mathbb{1}_{\{\tau_e<T\}}\middle|\mathcal{F}_{t_0}\right] \\
= & \int_{t_0}^{T}\left[\int_{\mathbb{R}}(\phi K-\phi S)^{+}\mathbb{Q}\left(S_T\in dS\middle|S_u=E_u\right)\right]\mathbb{Q}\left(\tau_e\in du\middle|\mathcal{F}_{t_0}\right) \\
= & \int_{t_0}^{T}\mathbb{E}_{\mathbb{Q}}\left[(\phi K-\phi S_T)^{+}\middle|S_u=E_u\right]\mathbb{Q}\left(\tau_e\in du\middle|\mathcal{F}_{t_0}\right). \qquad (15)
\end{aligned}
$$

Moreover, considering equation (8), the expectation contained in the right-hand side of equation (15) can be expressed in terms of a European option price:

$$
\mathbb{E}_{\mathbb{Q}}\left[(\phi K-\phi S_T)^{+}\mathbb{1}_{\{\tau_e<T\}}\middle|\mathcal{F}_{t_0}\right]=\int_{t_0}^{T}e^{r(T-u)}v_u\left(E,K,T;\phi\right)\mathbb{Q}\left(\tau_e\in du\middle|\mathcal{F}_{t_0}\right). \qquad (16)
$$

Finally, combining equations (9), (12) and (16), the early exercise representation (11) follows. ∎

Under the usual geometric Brownian motion assumption, equation (11) yields a closed-form solution to the early exercise premium (modulo to the specification of the first passage time density), because the term $v_u\left(E,K,T;\phi\right)$ can be computed using the Merton (1973) formulae. The same reasoning applies to the CEV model since, in this case, European option prices can be computed through the analytical solutions provided by Cox (1975) or Schroder (1989). Note, however, that the proof of Proposition 1 relies only on the much weaker assumption of a Markovian and diffusive asset price. That is, the early exercise representation (11) is still valid for other asset price processes beyond the general class represented by the stochastic differential equation (2).

The representation offered by Proposition 1 is also amenable to an intuitive interpretation. Since the *value-matching condition* implies that $(\phi K-\phi E_u)=V_u\left(E,K,T;\phi\right)$, then equation (11) can be rewritten as

$$
\begin{aligned}
& EEP_{t_0}\left(S,K,T;\phi\right) \\
= & \int_{t_0}^{T}e^{-r(u-t_0)}\left[V_u\left(E,K,T;\phi\right)-v_u\left(E,K,T;\phi\right)\right]\mathbb{Q}\left(\tau_e\in du\middle|\mathcal{F}_{t_0}\right).
\end{aligned}
$$

Using equation (10), today's early exercise premium can now be easily understood as the discounted expectation of the early exercise premium stopped at the first passage time:[4]

$$EEP_{t_0}(S,K,T;\phi) = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(\tau_e - t_0)} EEP_{\tau_e}(E,K,T;\phi)\, \mathbb{1}_{\{\tau_e < T\}} \middle| \mathcal{F}_{t_0}\right]. \tag{17}$$

That is, the discounted and stopped early exercise premium is, as expected, a martingale under measure $\mathbb{Q}$.[5]

Such an interpretation is substantially different from the one implicit in the characterization of the American option already offered by Kim (1990), Jacka (1991), Carr, Jarrow, and Myneni (1992), Kim and Yu (1996), and Detemple and Tian (2002). For all these authors, the early exercise premium corresponds to the compensation that the option holder would require (in the *stopping region*) in order to postpone exercise until the maturity date. Under the geometric Brownian motion assumption, and for some early exercise boundary specifications—see, for example, Ju (1998)—it is possible to obtain closed-form solutions for such early exercise representation. However, for more general underlying diffusion price processes, as the ones proposed by Kim and Yu (1996), and Detemple and Tian (2002), it is necessary to solve numerically and recursively a set of *value-matching* implicit integral equations, which can be too time-consuming for practical purposes. To improve efficiency, Huang, Subrahmanyam, and Yu (1996) calculate only option values based on a few points on an approximation to the exercise boundary, and then use Richardson extrapolation. Such accelerated recursive scheme is very fast but not very accurate, especially for medium- and long-term options—see, for example, Ju ((1998), Tables 1 and 2).

Alternatively, the new characterization offered by Proposition 1 can be efficiently applied for any early exercise boundary specification, and under any Markovian (and diffusion)

[4]It is well known that the discounted price process of an American option is a supermartingale under the risk-neutral measure. Nevertheless, such relative price process behaves as a martingale during any period of time in which it is not optimal to exercise the option. Therefore, the same result obtains until the first passage time to the exercise boundary.

[5]Alternatively and as suggested by an anonymous referee, the right-hand-side of equation (11) is simply the expected value of the cash flow that arises from liquidating (at the first passage time to the exercise boundary) a static portfolio that includes a long position on an American option and a short position on the corresponding European contract.

underlying price process, which constitutes an innovation with respect to the representations of the early exercise premium already offered in the literature.

B. Asymptotic Properties

Before implementing Proposition 1 and in order to investigate its limits, the asymptotic properties of the early exercise representation (11) are first explored.

Proposition 2 *Under the assumptions of Proposition 1, the early exercise premium and the American option value satisfy the following boundary conditions for $t \leq T$:*

$$\lim_{r \downarrow 0} EEP_t\left(S, K, T; 1\right) = 0, \tag{18}$$

$$V_T\left(S, K, T; \phi\right) = \left(\phi K - \phi S_T\right)^+, \tag{19}$$

$$\lim_{S \uparrow \infty} V_t\left(S, K, T; 1\right) = 0, \tag{20}$$

$$\lim_{S \downarrow 0} V_t\left(S, K, T; -1\right) = 0, \tag{21}$$

and

$$\lim_{S \to E_t} V_t\left(S, K, T; \phi\right) = \left(\phi K - \phi E_t\right), \tag{22}$$

where $\phi = -1$ for an American call or $\phi = 1$ for an American put.

Proof. See Appendix A. ∎

Once the general diffusion process (2) is adopted, the usual parabolic partial differential equation follows for the price of the American option.

Proposition 3 *Under the diffusion process (2), the American option value function given by Proposition 1 satisfies, for $\phi S_t > \phi E_t$ and $t \leq T$, the partial differential equation*

$$\mathcal{L} V_t\left(S, K, T; \phi\right) = 0, \tag{23}$$

where $\mathcal{L}$ is the parabolic operator

$$\mathcal{L} := \frac{\sigma(t,S)^2 S^2}{2}\frac{\partial^2}{\partial S^2} + (r-q) S \frac{\partial}{\partial S} - r + \frac{\partial}{\partial t}, \tag{24}$$

$\phi = -1$ for an American call and $\phi = 1$ for an American put.

Proof. See Appendix B. ∎

The relevance of Propositions 2 and 3 emerges from the fact that the American option price is, under the stochastic differential equation (2), the unique solution of the initial value problem represented by the partial differential equation (23) and by the boundary conditions (19) through (22).

Next proposition shows that the American option representation contained in Proposition 1 converges to the appropriate perpetual limit. This result contrasts with the characterization offered by Carr, Jarrow, and Myneni (1992) or Kim and Yu (1996), and can be relevant for the pricing of long-term option contracts. Explicit pricing solutions are also given for both the Merton (1973) and the CEV models, which will be used in the subsequent empirical analysis. The latter result constitutes an innovation with respect to the previous literature.

Proposition 4 *Under the geometric Brownian motion assumption, that is for $\sigma(t,S) = \sigma$ in equation (2), the American option value function given by Proposition 1 converges, in the limit, to the perpetual formulae given by McKean (1965) or Merton (1973), i.e.*

$$\lim_{T\uparrow\infty} V_t(S,K,T;\phi) = (\phi K - \phi E_\infty)\left(\frac{E_\infty}{S_t}\right)^{\gamma(\phi)}, \tag{25}$$

where $\phi S_t > \phi E_\infty$, E_∞ denotes the constant exercise boundary,

$$\gamma(\phi) := \frac{r - q - \frac{\sigma^2}{2} + \phi\sqrt{\left(r-q-\frac{\sigma^2}{2}\right)^2 + 2\sigma^2 r}}{\sigma^2}, \tag{26}$$

$\phi = -1$ for an American call and $\phi = 1$ for an American put.

Under the CEV model and for $r \neq q$, the perpetual American option price is equal to

$$\begin{aligned}\lim_{T\uparrow\infty} V_t(S,K,T;\phi) &= (\phi K - \phi E_\infty)\left(\frac{S_t}{E_\infty}\right)^{\eta(\phi)} \exp\{\eta(\phi)[x(S_t) - x(E_\infty)]\} \\ &\quad \frac{M_{\phi(\beta-2)}\left[\eta(\phi) + (-1)^{\eta(\phi)}\alpha, \frac{\beta-1-2\eta(\phi)}{\beta-2}; (-1)^{\eta(\phi)} x(S_t)\right]}{M_{\phi(\beta-2)}\left[\eta(\phi) + (-1)^{\eta(\phi)}\alpha, \frac{\beta-1-2\eta(\phi)}{\beta-2}; (-1)^{\eta(\phi)} x(E_\infty)\right]},\end{aligned} \tag{27}$$

where

$$\eta(\phi) := \begin{cases} \mathbb{1}_{\{r>q,\beta<2\}} \Leftarrow \phi = 1 \\ 1 - \mathbb{1}_{\{r>q,\beta>2\}} \Leftarrow \phi = -1 \end{cases}, \tag{28}$$

$$\alpha := \frac{r}{(\beta-2)(r-q)}, \tag{29}$$

$$x(S) := \frac{2(r-q)}{\delta^2(\beta-2)} S^{2-\beta}, \tag{30}$$

and

$$M_\lambda(a,b;z) := \begin{cases} M(a,b;z) \Leftarrow \lambda > 0 \\ U(a,b;z) \Leftarrow \lambda < 0 \end{cases}, \tag{31}$$

with $M(a,b;z)$ and $U(a,b;z)$ representing the confluent hypergeometric Kummer's functions.[6] *For $r = q$,*

$$\lim_{T\uparrow\infty} V_t(S,K,T;\phi) = (\phi K - \phi E_\infty)\sqrt{\frac{S_t}{E_\infty}} \frac{I_{\frac{1}{|\beta-2|};\phi(\beta-2)}\left[\varepsilon(S_t)\sqrt{2r}\right]}{I_{\frac{1}{|\beta-2|};\phi(\beta-2)}\left[\varepsilon(E_\infty)\sqrt{2r}\right]}, \tag{32}$$

where

$$\varepsilon(S) := \frac{2S^{1-\frac{\beta}{2}}}{\delta|\beta-2|}, \tag{33}$$

and

$$I_{\nu;\lambda}(z) := \begin{cases} I_\nu(z) \Leftarrow \lambda > 0 \\ K_\nu(z) \Leftarrow \lambda < 0 \end{cases}, \tag{34}$$

with $I_\nu(z)$ and $K_\nu(z)$ representing the modified Bessel functions.[7]

Proof. See Appendix C. ∎

[6] As defined by Abramowitz and Stegun ((1972), equations 13.1.2 and 13.1.3).

[7] See, for instance, Abramowitz and Stegun ((1972), p. 375).

IV. The First Passage Time Density

To implement the new American option value representation offered by Proposition 1, it is necessary to compute the first passage time density of the underlying asset price to the moving exercise boundary.

Following Buonocore, Nobile, and Ricciardi ((1987), eq. 2.7), a Fortet (1943)-type integral equation can be obtained for the optimal stopping time density under consideration. Notably, such non-linear integral equation involves only the transition density function of the underlying asset price. This result, contained in the next proposition, is valid for any Markovian underlying diffusion process and for any continuous representation of the exercise boundary.

Proposition 5 *Assuming that the underlying asset price S_t follows a Markovian diffusion process and considering that the optimal exercise boundary is a continuous function of time, the first passage time density of the underlying asset price to the moving exercise boundary is the implicit solution of the following non-linear integral equation:*

$$\int_{t_0}^{u} \mathbb{Q}\left(\phi S_u \leq \phi E_u \middle| S_v = E_v\right) \mathbb{Q}\left(\tau_e \in dv \middle| \mathcal{F}_{t_0}\right) = \mathbb{Q}\left(\phi S_u \leq \phi E_u \middle| \mathcal{F}_{t_0}\right), \tag{35}$$

for $\phi S_{t_0} > \phi E_{t_0}$, where $u \in [t_0, T]$, and with $\phi = -1$ for an American call or $\phi = 1$ for an American put.

Proof. Assuming that the exercise boundary is continuous on $[t_0, u]$ and that $\phi S_{t_0} > \phi E_{t_0}$, while using definition (5), the distribution function of the optimal stopping time can be written as:[8]

$$\begin{aligned} \mathbb{Q}\left(\tau_e \leq u \middle| \mathcal{F}_{t_0}\right) &= \mathbb{Q}\left[\inf_{t_0 \leq v \leq u}\left(\phi S_v - \phi E_v\right) \leq 0, \phi S_u \leq \phi E_u \middle| \mathcal{F}_{t_0}\right] \\ &\quad + \mathbb{Q}\left[\inf_{t_0 \leq v < u}\left(\phi S_v - \phi E_v\right) \leq 0, \phi S_u > \phi E_u \middle| \mathcal{F}_{t_0}\right]. \end{aligned}$$

[8] Notice that $\inf_{t_0 \leq v < u}\left[-\left(S_v - E_v\right)\right] = -\sup_{t_0 \leq v < u}\left(S_v - E_v\right)$.

Since $\mathbb{Q}\left[\inf_{t_0 \le v \le u}\left(\phi S_v - \phi E_v\right) \le 0, \phi S_u \le \phi E_u \middle| \mathcal{F}_{t_0}\right] = \mathbb{Q}\left(\phi S_u \le \phi E_u \middle| \mathcal{F}_{t_0}\right)$ and because the underlying price process is assumed to be Markovian,

$$\begin{aligned} (36) \qquad \mathbb{Q}\left(\tau_e \le u \middle| \mathcal{F}_{t_0}\right) &= \mathbb{Q}\left(\phi S_u \le \phi E_u \middle| \mathcal{F}_{t_0}\right) \\ &+ \int_{t_0}^{u} \mathbb{Q}\left(\phi S_u > \phi E_u \middle| S_v = E_v\right) \mathbb{Q}\left(\tau_e \in dv \middle| \mathcal{F}_{t_0}\right). \end{aligned}$$

Finally, considering that $\mathbb{Q}\left(\tau_e \le u \middle| \mathcal{F}_{t_0}\right) = \int_{t_0}^{u} \mathbb{Q}\left(\tau_e \in dv \middle| \mathcal{F}_{t_0}\right)$, equation (35) follows immediately from equation (36). ∎

Propositions 1 and 5 show that an explicit solution for the European option and knowledge of the transition density function of the underlying price process are the only requirements for the analytical valuation of the American contract. Hence, the proposed methodology can be fruitfully applied to many other Markovian pricing systems besides the standard case covered by equation (2). One of such extensions will be discussed in Section VII.

Proposition 5 can be specialized easily for the Merton (1973) and the CEV models, which will be used in the numerical analysis to be presented in Section VI. For $\sigma(t,S) = \sigma$, the underlying price process—as given by equation (2)—becomes lognormally distributed, and equation (35) can be restated as

$$(37) \qquad \int_{t_0}^{u} \Phi\left(\phi \frac{E_v^z - E_u^z}{\sqrt{u-v}}\right) \mathbb{Q}\left(\tau_e \in dv \middle| \mathcal{F}_{t_0}\right) = \Phi\left(-\phi \frac{E_u^z}{\sqrt{u-t_0}}\right),$$

with

$$(38) \qquad E_v^z := \frac{\ln\left(\frac{S_{t_0}}{E_v}\right) + \left(r - q - \frac{\sigma^2}{2}\right)(v - t_0)}{\sigma},$$

and where $\Phi(\cdot)$ represents the cumulative density function of the univariate standard normal distribution. Equation (37) is consistent with Park and Schuurmann ((1976), Theorem 1) and similar to the integral equation used by Longstaff and Schwartz ((1995), eq. A6). For $\sigma(t,S) = \delta S_t^{\frac{\beta}{2}-1}$, it is well known—see, for example, Schroder ((1989), eq. 1) for $\beta < 2$, or Emanuel and MacBeth ((1982), eq. 7) for $\beta > 2$—that

$$(39) \qquad \mathbb{Q}\left(S_u \le E_u \middle| S_v = E_v\right) = \begin{cases} Q_{\chi^2\left(\frac{2}{2-\beta}, 2\kappa E_u^{2-\beta}\right)}\left(2\kappa E_v^{2-\beta} e^{(2-\beta)(r-q)(u-v)}\right) \Leftarrow \beta < 2 \\ Q_{\chi^2\left(2+\frac{2}{\beta-2}, 2\kappa E_v^{2-\beta} e^{(2-\beta)(r-q)(u-v)}\right)}\left(2\kappa E_u^{2-\beta}\right) \Leftarrow \beta > 2 \end{cases},$$

with

$$\kappa := \frac{2(r-q)}{(2-\beta)\,\delta^2\left[e^{(2-\beta)(r-q)(u-v)}-1\right]}, \tag{40}$$

and where $Q_{\chi^2(a,b)}(x)$ represents the complementary distribution function of a non-central chi-square law with a degrees of freedom and non-centrality parameter b. Combining equations (35) and (39), a non-linear integral equation follows immediately for the optimal stopping time density under the CEV model.

Except for such crude critical asset price specifications as, for example, the constant and exponential functional forms used by Ingersoll (1998) under the geometric Brownian motion assumption, the optimal stopping time density is not known in closed-form. Following Kuan and Webber (2003), the next proposition shows that such first passage time density can be efficiently computed, for any exercise boundary specification, through the standard partition method proposed by Park and Schuurmann (1976).

Proposition 6 *Under the assumptions of Proposition 5, and dividing the time-interval* $[t_0, T]$ *into* N *sub-intervals of (equal) size* $h := \frac{T-t_0}{N}$*, then*

$$\begin{aligned} EEP_{t_0}(S,K,T;\phi) &= \sum_{i=1}^{N}\left\{\left[\phi K - \phi E_{t_0+\frac{(2i-1)h}{2}}\right] - v_{t_0+\frac{(2i-1)h}{2}}(E,K,T;\phi)\right\} \\ &\quad e^{-r\frac{(2i-1)h}{2}}\left[\mathbb{Q}(\tau_e = t_0+ih) - \mathbb{Q}(\tau_e = t_0+(i-1)h)\right], \end{aligned} \tag{41}$$

where $\phi = -1$ *for an American call or* $\phi = 1$ *for an American put. The probabilities* $\mathbb{Q}(\tau_e = t_0 + ih)$ *are obtained from the following recurrence relation:*

$$\begin{aligned} &\mathbb{Q}(\tau_e = t_0+ih) \\ = \;&\mathbb{Q}(\tau_e = t_0+(i-1)h) + \left\{F_\phi\left[E_{t_0+ih}; E_{t_0+\frac{(2i-1)h}{2}}\right]\right\}^{-1} \\ &\left\{F_\phi\left(E_{t_0+ih}; S_{t_0}\right) - \sum_{j=1}^{i-1} F_\phi\left[E_{t_0+ih}; E_{t_0+\frac{(2j-1)h}{2}}\right]\right. \\ &\left.\left[\mathbb{Q}(\tau_e = t_0+jh) - \mathbb{Q}(\tau_e = t_0+(j-1)h)\right]\right\}, \end{aligned} \tag{42}$$

for $i = 1, \ldots, N$*, where* $\mathbb{Q}(\tau_e = t_0) = 0$*, and with*

$$F_\phi\left(E_u; S_{t_0}\right) := \mathbb{Q}\left(\phi S_u \le \phi E_u \middle| \mathcal{F}_{t_0}\right) \tag{43}$$

representing the risk-neutral cumulative density function, for $\phi = 1$, or the complementary distribution function, for $\phi = -1$, of the underlying price process.

Proof. Equations (41) and (42) are obtained via discretization of equations (11) and (35) for the partition $t_0 < t_1 < \ldots < t_N = T$, where $t_i = t_0 + ih$ $(i = 1, \ldots, N)$, and $u = \frac{t_i + t_{i-1}}{2}$. ∎

V. Specification of the Exercise Boundary

The pricing solution offered by Proposition 1 depends on the specification adopted for the exercise boundary $\{E_t, t_0 \leq t \leq T\}$. Although such an optimal exercise policy is not known *ex ante* (i.e., before the solution of the pricing problem), its main characteristics have already been established in the literature:[9] *i*) The exercise boundary is a continuous function of time—see, for instance, Jacka ((1991), Propositions 2.2.4 and 2.2.5); *ii*) E_t is a non-decreasing function of time t for the American put, but non-increasing for the American call contract—see Jacka ((1991), Proposition 2.2.2); *iii*) the exercise boundary is limited by $E_T = \phi \min\left(\phi K, \phi \frac{r}{q} K\right)$—as stated in Van Moerbeke (1976); and *iv*) $\lim_{t \uparrow \infty} E_t = E_\infty$, where E_∞ represents the (constant) critical asset price for the perpetual American case.

As described by Ingersoll ((1998), p. 89), in order to price an American option, it is necessary to choose a parametric family $\mathcal{E}$ of exercise policies $E_t\left(\underline{\theta}\right)$, where each policy is characterized by an n-dimensional vector of parameters $\underline{\theta} \in \mathbb{R}^n$. Then, the early exercise value (as given by equation (11)) is expressed as a function of $\underline{\theta}$ and maximized with respect to the parameters. Since the chosen family $\mathcal{E}$ may not contain the optimal exercise boundary, the resulting American option price constitutes a lower bound for the true option value.

Of course, the more general the specification adopted for the exercise boundary, the smaller the approximation error associated with the American price estimate should be. However, the parametric families already proposed in the literature have been chosen not for their generality but because they provide fast analytical pricing solutions. In order to

[9] Bunch and Johnson (2000) propose, under the Merton (1973) model, an approximation for the critical-stock-price function which is accurate for small times to maturity.

measure the accuracy improvement provided by more general families of exercise policies, Section VI will consider the following parametric specifications:

1. Constant exercise boundary:

   (44) $$E_t\left(\underline{\theta}\right) = \theta_1, \; \theta_1 > 0.$$

   This is the simplest specification one can adopt and has already been used by Ingersoll (1998) and Sbuelz (2004), under the geometric Brownian motion assumption. Although it yields a closed-form solution for equation (11), such an exercise boundary cannot simultaneously satisfy previously stated requirements (iii) and (iv).

2. Exponential family:

   (45) $$E_t\left(\underline{\theta}\right) = \theta_1 e^{\theta_2 (T-t)}, \; \theta_1 > 0, \phi\theta_2 < 0.$$

   This specification, already proposed by Ingersoll (1998) for the geometric Brownian motion process, also yields an analytical solution for equation (11), but again cannot simultaneously satisfy requirements (iii) and (iv).

3. Exponential-constant family:

   (46) $$E_t\left(\underline{\theta}\right) = \theta_1 + e^{\theta_2 (T-t)}, \; \phi\theta_2 < 0.$$

   This new parameterization corresponds to a simple modification of equation (45) and has never been proposed in the literature. Section VI will show that it can produce smaller pricing errors than equation (45) for the same number of parameters.

4. Polynomial family:

   (47) $$E_t\left(\underline{\theta}\right) = \sum_{i=1}^{n} \theta_i \left(T-t\right)^{i-1}.$$

Because the exercise boundary is assumed to be continuous and defined on the closed interval $[t_0, T]$, the Weierstrass approximation theorem implies that E_t can be uniformly approximated, for any desired accuracy level, by the polynomial (47). By increasing the degree of the polynomial (and therefore, the number of parameters to be estimated), this new class of exercise policies allows the pricing error to be arbitrarily reduced. Section VI will reveal that with only five parameters it is possible to obtain smaller pricing errors than with many alternative specifications already proposed in the literature.

5. CJM family:

$$E_t\left(\underline{\theta}\right) = \phi \min\left(\phi K, \phi \frac{r}{q} K\right) e^{-\theta_1 \sqrt{T-t}} + E_\infty \left(1 - e^{-\theta_1 \sqrt{T-t}}\right), \ \theta_1 \geq 0. \tag{48}$$

Equation (48) corresponds to an exponentially weighted average between the terminal bound and the perpetual limit of the exercise boundary, and fulfills all of requirements (i)–(iv). Such a specification was proposed by Carr, Jarrow, and Myneni ((1992), p. 93), but has never been tested since it does not yield an analytical solution for the American option price. The next section will show that, with only one parameter, the magnitude of pricing errors produced by this specification is similar to that associated with the best parameterizations already available in the literature.

VI. Numerical Results

To test the accuracy and efficiency of the pricing solutions proposed in Proposition 1 and the influence of the exercise boundary specification on the early exercise value, all the parametric families described in Section V will be compared for different constellations of the coefficients contained in equation (2), and under two special cases: the geometric Brownian motion and the CEV processes. For this purpose, the maximization of the early exercise value (with respect to the parameters defining the exercise policy) is implemented through Powell's method, as described in Press, Flannery, Teukolsky, and Vetterling ((1994), Section 10.5).[10]

To enhance the efficiency of the proposed valuation method, the parameters defining the exercise policy are first estimated by discretizing both Propositions 1 and 5 using only $N = 2^4$ time-steps. Then, and based on such an approximation for the optimal exercise boundary, the early exercise premium is computed from Proposition 6 using $N = 2^8$ time steps. The crude discretization adopted in the optimization stage should not compromise the accuracy of the pricing formulae proposed because, as noted by Ju ((1998), p. 642) in the context of the Merton (1973) model, a detailed description of the early exercise boundary is not necessary to generate accurate American option values.

Table 1 values short maturity American put options under different specifications of the exercise boundary, and based on the option parameters contained in Broadie and Detemple ((1996), Table 1), and Ju ((1998), Table 1) for the Black and Scholes (1973) model. Accuracy is measured by the average absolute percentage error (over the 20 contracts considered) of each valuation approach and with respect to the exact American option price. This proxy of the "true" American put value (fourth column) is computed through the binomial tree model with $15,000$ time steps, as suggested by Broadie and Detemple ((1996), p. 1222). Efficiency is evaluated by the total CPU time (expressed in seconds) spent to value the whole set of contracts considered. All computations were made with Pascal programs running on an Intel Pentium 4 2.80 GHz processor under a Linux operating system.

Insert Table 1 about here.

The American put prices produced by the analytical pricing solutions associated with the constant and exponential boundary specifications (fifth and sixth columns of Table 1), as

[10]This method requires evaluations only of the function to be maximized and therefore is faster than a *conjugate gradient* or a *quasi-Newton* algorithm. Nevertheless, it is always possible to use a more robust optimization method, because the derivatives of the first passage time density can be computed through a recurrence relation similar to equation (42). Details are available upon request.

given by equations (44) and (45), respectively, are obtained from Ingersoll ((1998), sections 4 and 5). All the other early exercise boundary approximations (i.e., from the seventh to the tenth columns of Table 1) are implemented through Proposition 6. For comparison purposes, the last three columns of Table 1 contain the American put prices generated by the full (with $2,000$ time steps)[11] and the 10-point accelerated recursive methods of Huang, Subrahmanyam, and Yu (1996), and by the three-point multipiece exponential function method proposed by Ju (1998). The choice of the multipiece exponential approximation as a benchmark for the best pricing methods already proposed in the literature, under the geometric Brownian motion assumption, follows from Ju ((1998), Tables 3 and 5): it is faster than the randomization method of Carr (1998) (for the same accuracy level) and much more accurate, for hedging purposes, than the econometric approach of Broadie and Detemple (1996).

The fastest approximations (in terms of CPU time) are the constant, the exponential, and the three-point multipiece exponential specifications, as well as the accelerated recursive method of Huang, Subrahmanyam, and Yu (1996): they all possess computational times below 0.2 seconds for the range of all contracts under consideration. However, the pricing errors generated by the constant and the exponential parameterizations can be significant. For instance, the average mispricing of the constant parameterization equals 41 basis points. Additionally, and as shown by Ju ((1998), Tables 1 and 2), the accuracy of the 10-point recursive scheme deteriorates as the option maturity increases.

With the same number of parameters as the already known exponential approximation, the new exponential-constant parameterization can yield pricing errors about three times smaller. Even more interestingly, the CJM approximation suggested by Carr, Jarrow, and Myneni (1992) and tested here possesses an accuracy similar to the three-point multipiece exponential approach. This result is relevant since the CJM approximation satisfies all the requirements described in Section V for the early exercise boundary specification.

Table 1 also shows that the implementation of a polynomial approximation is able to achieve smaller pricing errors than the Ju (1998) approach. The Huang, Subrahmanyam,

[11] As suggested by Detemple and Tian ((2002), p. 924).

and Yu (1996) full recursive method yields an even higher precision level, but at the expense of a prohibitive computational effort. Overall, taking into consideration both accuracy and efficiency, the best pricing methodology, under the geometric Brownian motion assumption, is still the multipiece exponential approach of Ju (1998). Nevertheless, the disparity of pricing errors contained in Table 1 shows that the early exercise premium depends largely on the specification adopted for the early exercise boundary.

Insert Table 2 about here.

Tables 2 and 3 repeat the analysis contained in Table 1 for the same parameter values, but under the CEV model. Table 2 assumes $\beta = 3$ (> 2) and prices American put contracts with a time-to-maturity of six months, while Table 3 considers a square root process with $\beta = 1$ (< 2) and American call options with a time-to-maturity of one year. Parameter δ is computed from equation (3) by imposing the same instantaneous volatility as in Table 1.

The proxy of the exact American option price (fourth column) is now computed through the Crank-Nicolson finite difference method with $15,000$ time intervals and $10,000$ space steps. Besides the early exercise boundary specifications described in Section V, Tables 2 and 3 also contain the full recursive scheme (eleventh column), as suggested by Detemple and Tian ((2002), Proposition 3), and a 10-point accelerated recursive approach (last column), along the lines of Kim and Yu ((1996), subsection 3.4).[12]

Insert Table 3 about here.

As before, the constant specification generates excessively large (absolute) pricing errors and the new exponential-constant parameterization yields an accuracy higher than the

[12]The trinomial approach developed by Boyle and Tian (1999) for the valuation of barrier and lookback options under the CEV model (for $0 \leq \beta < 2$) can also be used to price American standard calls and puts. However, the numerical experiments run have shown that the adopted Crank-Nicolson scheme possesses better convergence properties.

exponential specification for American put contracts (see Table 2). In contrast, Table 3 shows that the exponential boundary is more accurate for American call contracts than the new formulation given by equation (46). Under the CEV model, the CJM approximation presents an excellent performance even though the pricing errors are now affected by the approximation employed to evaluate the non-central chi-square distribution function,[13] as well as by the root-finding routine used to extract the optimal constant exercise boundary E_∞ from equations (27) and (32).

In terms of accuracy, the Detemple and Tian (2002) approach constitutes the best pricing method for the CEV model. However, this approach is based on the full recursive method (with $2,000$ time steps) of Huang, Subrahmanyam, and Yu (1996), which is very time consuming—six times slower than the exact Crank-Nicolson implicit finite-difference scheme. The accelerated recursive scheme of Kim and Yu (1996) is much more efficient but can also be inaccurate for medium- and long-term options. The last column of Table 3 shows a mean absolute percentage error of about 16 basis points. On the contrary, Tables 1 through 3 show that the accuracy of the pricing methodology proposed in Proposition 1 is not affected by the time-to-maturity of the option contract under valuation. Moreover, for almost all the parameterizations tested (with the single exception of the polynomial specification), the computational time of the proposed pricing methodology corresponds to less than one second per contract.

Insert Table 4 about here.

Under the CEV model, the best trade-off between accuracy and efficiency is given by the polynomial approximations presented in Tables 2 and 3, since their accuracy can always be improved by increasing their degree. Table 4 applies different polynomial specifications to a

[13] Equation (39) is computed from routine "cumchn", which is contained in the Fortran library of Brown, Lovato, and Russell (1997). This routine is based on Abramowitz and Stegun ((1972), eq. 26.4.25), and is found to be more precise than the algorithm offered by Schroder (1989) or the *Wiener germ* approximations proposed by Penev and Raykov (1997), especially for large values of the non-centrality parameter or of the upper integration limit.

random sample of $1,250$ American put options, where all the option parameters, with the exception of β and δ, are extracted from the same uniform distributions as in Ju ((1998), Table 3).[14] With a six-degree polynomial it is possible to obtain an average absolute percentage error (computed against the Crank-Nicolson solution) of only 1.5 basis points and a maximum absolute percentage error of about 9 basis points, which corresponds to a higher accuracy than that associated with the 10-point accelerated recursive scheme.

As expected, the pricing errors produced by the specifications described in Section V are negative because any approximation of the optimal exercise policy can yield only a lower bound for the true American option price. The only exception corresponds to the 10-point recursive method, which might be explained by the non-uniform convergence of the Richardson extrapolation employed.

In summary, the numerical results presented in Tables 2, 3 and 4 configure the implementation of Proposition 1 through a polynomial specification of the early exercise boundary as the best pricing alternative, under the CEV model, for medium- and long-term American option contracts.

VII. Extension to Credit Risk Modeling

This section shows that the optimal stopping approach proposed in this paper is easily extended to the context of the Carr and Linetsky (2006) model, yielding analytical pricing solutions for American equity options under default risk.

Carr and Linetsky (2006) construct a unified framework for the valuation of corporate liabilities, credit derivatives, and equity derivatives as contingent claims written on a default-

[14]From the uniform distribution adopted for the instantaneous volatility, the parameter δ is obtained from equation (3). Parameter β is assumed to possess a uniform distribution between 0 and 4.0. The scenario $\beta < 0$ is ignored because it would imply unrealistic economic properties for the CEV process; namely, bankruptcy would be attainable for sufficiently negative values of β (which is implausible, for instance, when considering options on stock indices), and underlying asset price volatility would explode as the spot price tends to the origin.

able stock. The price of the defaultable stock is modeled as a time-inhomogeneous diffusion process solving the stochastic differential equation

$$\frac{dS_t}{S_t} = [r_t - q_t + \lambda(t, S)]\, dt + \sigma(t, S)\, dW_t^{\mathbb{Q}}, \tag{49}$$

with $S_{t_0} > 0$, and where the interest rate r_t and the dividend yield q_t are now deterministic functions of time, while the instantaneous volatility of equity returns $\sigma(t, S)$ and the default intensity $\lambda(t, S)$ can also be state-dependent. Again, $\mathbb{F} = \{\mathcal{F}_t : t \geq t_0\}$ is the filtration generated by the standard Wiener process $W_t^{\mathbb{Q}} \in \mathbb{R}$, and the equivalent martingale measure $\mathbb{Q}$ is taken as given.[15]

The pricing model proposed by Carr and Linetsky (2006) can either diffuse or jump to default. In the first case, bankruptcy occurs at the first passage time of the stock price to zero:

$$\tau_0 := \inf\{t > t_0 : S_t = 0\}. \tag{50}$$

Alternatively, the stock price can also jump to a *cemetery state* whenever the hazard process $\frac{1}{\mathbb{1}_{\{t<\tau_0\}}}\int_{t_0}^{t} \lambda(u, S)\, du$ is greater or equal to the level drawn from an exponential random variable Θ independent of $W_t^{\mathbb{Q}}$ and with unit mean, i.e. at the first jump time

$$\tilde{\zeta} := \inf\left\{t > t_0 : \frac{1}{\mathbb{1}_{\{t<\tau_0\}}}\int_{t_0}^{t} \lambda(u, S)\, du \geq \Theta\right\} \tag{51}$$

of a doubly-stochastic Poisson process with intensity $\lambda(t, S)$. Therefore, the time of default is simply given by

$$\zeta = \tau_0 \wedge \tilde{\zeta}, \tag{52}$$

and $\mathbb{D} = \{\mathcal{D}_t : t \geq t_0\}$ is the filtration generated by the default indicator process $\mathcal{D}_t = \mathbb{1}_{\{t>\zeta\}}$.

[15] The inclusion of the hazard rate $\lambda(t, S)$ in the drift of equation (49) compensates the stockholders for default (with zero recovery) and insures, under measure $\mathbb{Q}$, an expected rate of return equal to the risk-free interest rate. Nevertheless, such equivalent martingale measure will not be unique because the arbitrage-free market considered by Carr and Linetsky (2006) is incomplete in the sense that the jump to default will not be modeled as a stopping time of $\mathbb{F}$.

Using the same terminology as in Section II, the time-t_0 value of an American option on the stock price S, with strike price K, and with maturity date T can now be represented by the following Snell envelope:

$$
\begin{aligned}
(53) \qquad V_{t_0}(S,K,T;\phi) &= \sup_{\tau\in\mathcal{T}}\left\{\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{T\wedge\tau} r_l dl}\left(\phi K-\phi S_{T\wedge\tau}\right)^{+}\mathbb{1}_{\{\zeta>T\wedge\tau\}}\middle|\mathcal{G}_{t_0}\right]\right. \\
&\quad \left.+\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\zeta} r_l dl}\left(\phi K\right)^{+}\mathbb{1}_{\{\zeta\le T\wedge\tau\}}\middle|\mathcal{G}_{t_0}\right]\right\},
\end{aligned}
$$

where $\mathcal{T}$ is the set of all stopping times (taking values in $[t_0,\infty]$) for the enlarged filtration $\mathbb{G}=\{\mathcal{G}_t : t\ge t_0\}$, with $\mathcal{G}_t=\mathcal{F}_t\vee\mathcal{D}_t$. For the American call ($\phi=-1$) there is no recovery if the firm defaults. However, for the American put ($\phi=1$), the second expectation on the right-hand side of equation (53) corresponds to a recovery payment equal to the strike K at the default time $\zeta\le T\wedge\tau$. Moreover, since the (unknown) early exercise boundary lies between zero (the bankruptcy boundary) and S_{t_0} (given that the American put is assumed to be alive on the valuation date), then the default event cannot precede the early exercise of the option contract, that is $\{\zeta\le T\wedge\tau\}=\{\zeta\le T\}$. Therefore,

$$
(54) \qquad V_{t_0}(S,K,T;\phi)=V_{t_0}^{0}(S,K,T;\phi)+V_{t_0}^{D}(S,K,T;\phi),
$$

where

$$
\begin{aligned}
(55) \qquad V_{t_0}^{0}(S,K,T;\phi) &= \sup_{\tau\in\mathcal{T}}\left\{\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\tau} r_l dl}\left(\phi K-\phi S_{\tau}\right)^{+}\mathbb{1}_{\{\zeta>\tau\}}\mathbb{1}_{\{\tau<T\}}\middle|\mathcal{G}_{t_0}\right]\right. \\
&\quad \left.+e^{-\int_{t_0}^{T} r_l dl}\mathbb{E}_{\mathbb{Q}}\left[\left(\phi K-\phi S_{T}\right)^{+}\mathbb{1}_{\{\zeta>T\}}\mathbb{1}_{\{\tau\ge T\}}\middle|\mathcal{G}_{t_0}\right]\right\},
\end{aligned}
$$

and

$$
(56) \qquad V_{t_0}^{D}(S,K,T;\phi)=(\phi K)^{+}\,\mathbb{E}_{\mathbb{Q}}\left(e^{-\int_{t_0}^{\zeta} r_l dl}\mathbb{1}_{\{\zeta\le T\}}\middle|\mathcal{G}_{t_0}\right).
$$

For the American put, the term $V_{t_0}^{D}(S,K,T;1)$ is essentially an American-style default-contingent claim, which is similar to the floating leg of a credit default swap (CDS), and can be valued through Carr and Linetsky ((2006), eq. 3.4). Concerning the American option value conditional on no default, identity (7) implies that

$$
(57) \qquad V_{t_0}^{0}(S,K,T;\phi)=v_{t_0}^{0}(S,K,T;\phi)+EEP_{t_0}^{0}(S,K,T;\phi),
$$

where

$$v_{t_0}^0(S,K,T;\phi) = e^{-\int_{t_0}^T r_l dl}\mathbb{E}_{\mathbb{Q}}\left[(\phi K - \phi S_T)^+ \mathbb{1}_{\{\zeta>T\}}\middle|\mathcal{G}_{t_0}\right] \tag{58}$$

represents the time-t_0 price of the corresponding European option (conditional on no default until the maturity date T), and the early exercise premium is equal to

$$\begin{aligned} EEP_{t_0}^0(S,K,T;\phi) &= \sup_{\tau\in\mathcal{T}}\left\{\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^\tau r_l dl}(\phi K-\phi S_\tau)^+\mathbb{1}_{\{\zeta>\tau\}}\mathbb{1}_{\{\tau<T\}}\middle|\mathcal{G}_{t_0}\right]\right. \\ &\quad \left. -e^{-\int_{t_0}^T r_l dl}\mathbb{E}_{\mathbb{Q}}\left[(\phi K-\phi S_T)^+\mathbb{1}_{\{\zeta>T\}}\mathbb{1}_{\{\tau<T\}}\middle|\mathcal{G}_{t_0}\right]\right\}. \end{aligned} \tag{59}$$

The next proposition provides an analytical representation of the early exercise premium given in equation (59).

Proposition 7 *Under the pricing model defined by equations (49) through (52), the time-t_0 value of the early exercise premium for an American option on the stock price S, with strike price K, and with maturity date T is equal to*

$$\begin{aligned} & EEP_{t_0}^0(S,K,T;\phi) \\ &= \int_{t_0}^T e^{-\int_{t_0}^u r_l dl}\left[(\phi K-\phi E_u)^+ - v_u^0(E,K,T;\phi)\right] SP(t_0,u)\,\mathbb{Q}\left(\tau_e\in du\middle|\mathcal{F}_{t_0}\right), \end{aligned} \tag{60}$$

where $\{E_u, t_0\le u\le T\}$ is the (unknown) early exercise boundary, $\phi=1$ (-1) for an American put (call), and

$$SP(t_0,u) := \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^u \lambda(l,S)dl}\mathbb{1}_{\{\tau_0>u\}}\middle|\mathcal{F}_{t_0}\right] \tag{61}$$

represents the risk-neutral probability of surviving beyond time $u>t_0$. The first passage time τ_e is defined by equation (5), and its probability density function is still recovered through equation (35).

Proof. See Appendix D. ∎

Because both the default intensity and the instantaneous stock volatility have been left unspecified, Proposition 7 can be applied to many defaultable stock models already available in the literature, such as those proposed by Madan and Unal (1998) or Linetsky (2006).

Carr and Linetsky (2006) try to accommodate the leverage effect by adopting a CEV specification for the instantaneous stock volatility:

$$\sigma(t, S) = a_t S_t^{\bar{\beta}}, \tag{62}$$

where $\bar{\beta} < 0$ is the volatility elasticity parameter, and $a_t > 0$ is a deterministic volatility scale function. To be consistent with the empirical evidence of a positive relationship between default probabilities and equity volatility, Carr and Linetsky (2006) further assume that the default intensity is an increasing affine function of the instantaneous stock variance:

$$\lambda(t, S) = b_t + c\sigma(t, S)^2, \tag{63}$$

where $c \geq 0$, and $b_t \geq 0$ is a deterministic function of time. In summary, equations (49) to (52), together with equations (62) and (63) constitute the *jump to default extended CEV* model (JDCEV) proposed by Carr and Linetsky (2006).[16]

Compared with the previous literature on defaultable stock models, the JDCEV model offers an exact match to the term structures of CDS spreads and/or at-the-money implied volatilities (through the time-dependent functions a_t and b_t), even though it explicitly incorporates the dependency on the current stock price S of both $\lambda(t, S)$ and $\sigma(t, S)$. Nevertheless, the JDCEV model preserves analytical tractability since it offers closed-form solutions for both European options and the transition density function of the underlying stock price process. The next proposition provides an analytical solution for the distribution function of the price process S, which allows the first passage time density to be determined for the JDCEV model through the numerical solution of the non-linear integral equation (35).

[16] Note that the time-homogeneous version of the JDCEV model with $b = c = 0$ is reduced to a CEV process with absorption at point 0. In this case, equations (11) and (60) differ only because the former takes the survival probability to equal 1. However, since Delbaen and Shirakawa ((2002), Theorem 4.2) have shown that the CEV model admits arbitrage opportunities when it is conditioned to be strictly positive, Proposition 7 should also be applied to the CEV process.

Proposition 8 *Under the JDCEV model defined by equations (49) through (52), (62) and (63),*

$$\text{(64)} \qquad \mathbb{Q}\left(S_u \leq E_u \middle| S_l = E_l\right)$$
$$= \left[\frac{k(l,l;E_l)^2}{\tau(l,u)}\right]^{(-v)^+} \Phi^- \left[-(-v)^+, \frac{k(l,u;E_u)^2}{\tau(l,u)}; 2(1+|v|), \frac{k(l,l;E_l)^2}{\tau(l,u)}\right],$$

where $l \leq u$,

$$\text{(65)} \qquad v := \frac{c-\frac{1}{2}}{\left|\bar{\beta}\right|},$$

$$\text{(66)} \qquad \tau(l,u) := \int_l^u a_s^2 e^{-2\left|\bar{\beta}\right| \int_l^s \alpha_h dh} ds$$

with

$$\text{(67)} \qquad \alpha_u := r_u - q_u + b_u,$$

$$\text{(68)} \qquad k(l,u;E) := \frac{1}{\left|\bar{\beta}\right|} E^{\left|\bar{\beta}\right|} e^{-\left|\bar{\beta}\right| \int_l^u \alpha_s ds}$$

for any $E \in \mathbb{R}$, and $\Phi^-(p,y;a,b) := \mathbb{E}\left(X^p \mathbb{1}_{\{X \leq y\}}\right)$ represents the truncated p-th moment of a non-central chi-square random variable X with a degrees of freedom and non-centrality parameter b.

Proof. See Appendix E. ∎

To illustrate the proposed pricing methodology, Table 5 prices long-term American put options based on the same parameter constellations used in Table 3, but under a time-homogeneous JDCEV model with $c = 0.5$ and $b = 0.02$. The European put prices conditional on no default—as given by equation (58) with $\phi = 1$—are computed through Carr and Linetsky ((2006), eq. 5.18), whereas the recovery component of the European put contract (fourth column of Table 5) is obtained from Carr and Linetsky ((2006), equations 3.11 and 5.14). For this purpose, the series solution provided by Carr and Linetsky ((2006), Lemma 5.1) to the function $\Phi^-(\cdot)$ is implemented using the Temme (1994) algorithm for the incomplete Gamma function. The early exercise premium is obtained from Propositions

7 and 8, under a four-degree polynomial specification, and where the risk-neutral survival probability (61) is recovered from Carr and Linetsky ((2006), eq. 5.14). Finally, the recovery component (56) of the American put contract (eighth column of Table 5) is computed by solving Carr and Linetsky ((2006), eq. 5.15) through Romberg's integration method on an open interval.

Insert Table 5 about here.

As already shown by Carr and Linetsky ((2006), Table 1) in the context of European put contracts, the value of the out-of-the-money American put options contained in Table 5 is largely dominated by the recovery component (56).

VIII. Conclusions

The main theoretical contribution of this paper consists in deriving an alternative characterization of the early exercise premium, which possesses appropriate asymptotic properties, and is valid for any Markovian and diffusion representation of the underlying asset price as well as for any parameterization of the exercise boundary. The proposed pricing methodology was also extended, in Proposition 7, to the valuation of American equity options under default risk, and specialized, in Proposition 8, to the context of the JDCEV model.

The intuitive representation offered by Proposition 1 is simply based on the observation that the discounted and stopped early exercise premium must be a martingale under the risk-neutral measure. Additionally, the Markov property ensures analytical tractability since it enables the decomposition of the joint density between the first hitting time and the underlying asset price through the convolution of their marginal densities.

To test the proposed pricing methodology and to highlight its generality, several parameterizations of the exercise boundary were compared under the geometric Brownian motion assumption and for the CEV process. For both option pricing models, the continuity of the early exercise boundary allows the pricing errors to be arbitrarily reduced through a polynomial specification, which can be easily accommodated by the proposed methodology.

Under the Merton (1973) model, the multipiece exponential approach of Ju (1998) offers the best compromise between accuracy and efficiency. However, under the CEV model, Proposition 1 provides the best pricing alternative for medium- and long-term American options. Whereas the early exercise premium formula proposed in equation (11) involves only a single time-integral, the representations offered by Kim and Yu (1996) or Detemple and Tian (2002) pose a more demanding two-dimensional integration problem (with respect to time and to the underlying asset price). Moreover, although Proposition 1 requires the numerical evaluation of the first passage time density (which is shown to be easily recovered from the transition density function), the formulas offered by Kim and Yu (1996), and Detemple and Tian (2002) rely on the numerical and recursive solution of a set of value-matching (or high-contact) implicit integral equations, which are too time-consuming for practical purposes. And, even though such a recursive scheme can be accelerated through Richardson extrapolation, the pricing methodology proposed by Huang, Subrahmanyam, and Yu (1996) may yield inaccurate results for medium- and long-term options.

Since the analytical pricing of American options under the geometric Brownian motion process is already well established through the randomization approach of Carr (1998) or the multipiece exponential boundary approximation of Ju (1998), the characterization proposed in Proposition 1 can be more fruitfully applied under alternative (but Markovian) stochastic processes for the underlying asset price, as exemplified, in this paper, by the CEV and JDCEV models. For this purpose to be accomplished in an efficient way, it is required only that the selected price process provides a viable valuation method for European options and for its transition density function. This should be the case for multivariate Markovian models accommodating stochastic volatility and/or stochastic interest rates, for which the recursive scheme of Kim and Yu (1996) cannot be applied. Nevertheless, given space constraints, both extensions are left for future work.

Appendix A. Proof of Proposition 2

Concerning the boundary condition (18), since

$$\begin{aligned}\lim_{r\downarrow 0} E_T &= \min\left(K, \lim_{r\downarrow 0}\frac{r}{q}K\right)\\ &= 0,\end{aligned}$$

and because the exercise boundary $\{E_u, t \leq u \leq T\}$ is a non-decreasing function of u for an American put, then

$$\lim_{r\downarrow 0} E_u = 0,\ \forall u \in [t,T]\,. \tag{A-1}$$

Combining equations (11) and (A-1),

$$\lim_{r\downarrow 0} EEP_t\,(S,K,T;1) = \int_t^T \left[K - \lim_{r\downarrow 0} v_u\,(0,K,T;1)\right] \lim_{r\downarrow 0}\mathbb{Q}\left(\tau_e \in du\middle|\,\mathcal{F}_t\right). \tag{A-2}$$

Finally, since $\left[e^{-r(T-u)}K - S_u\right]^+ \leq v_u\,(S,K,T;1) \leq e^{-r(T-u)}K$ follows from straightforward no-arbitrage arguments, then $\lim_{r\downarrow 0} v_u\,(0,K,T;1) = K$ and equation (A-2) yields the boundary condition (18).

The terminal condition (19) follows immediately from equation (10) because $v_T\,(S,K,T;\phi)$ $= (\phi K - \phi S_T)^+$ and $EEP_T\,(S,K,T;\phi) = 0$.

Concerning the boundary condition (20), because $\lim_{S\uparrow\infty} v_t\,(S,K,T;1) = 0$, equations (10) and (11) imply that:

$$\begin{aligned}&\lim_{S\uparrow\infty} V_t\,(S,K,T;1)\\ &= \int_t^T e^{-r(u-t)}\left[(K - E_u) - v_u\,(E,K,T;1)\right]\lim_{S\uparrow\infty}\mathbb{Q}\left(\tau_e \in du\middle|\,\mathcal{F}_t\right).\end{aligned} \tag{A-3}$$

Assuming that $\lim_{S\uparrow\infty} S_u = \infty,\ \forall u \geq t$, then $\lim_{S\uparrow\infty}\mathbb{Q}\left(\tau_e \in du\middle|\,\mathcal{F}_t\right) = 0$, and the boundary condition (20) is obtained, because the exercise boundary is independent of the current asset price and finite. Similar reasoning can be applied to derive the boundary condition (21).

Finally, the *value-matching condition* (22) is also easily derived from equations (10) and (11):

(A-4)
$$\begin{aligned}
& \lim_{S \to E_t} V_t(S, K, T; \phi) \\
= \; & v_t(E, K, T; \phi) \\
& + \int_t^T e^{-r(u-t)} \left[(\phi K - \phi E_u) - v_u(E, K, T; \phi) \right] \lim_{S \to E_t} \mathbb{Q}\left(\tau_e \in du \middle| \mathcal{F}_t \right).
\end{aligned}$$

Since

$$\lim_{S \to E_t} \mathbb{Q}\left(\tau_e \in du \middle| \mathcal{F}_{t_0} \right) = \delta(u - t),$$

where $\delta(\cdot)$ is the Dirac-delta function, equation (A-4) yields

$$\begin{aligned}
\lim_{S \to E_t} V_t(S, K, T; \phi) &= v_t(E, K, T; \phi) + e^{-r(t-t)} \left[(\phi K - \phi E_t) - v_t(E, K, T; \phi) \right] \\
&= (\phi K - \phi E_t).
\end{aligned}$$

∎

Appendix B. Proof of Proposition 3

Applying the parabolic operator $\mathcal{L}$ to equations (10) and (11), and using Leibniz's rule,

(B-1)
$$\begin{aligned}
& \mathcal{L} V_t(S, K, T; \phi) \\
= \; & \mathcal{L} v_t(S, K, T; \phi) + \int_t^T r e^{-r(u-t)} \left[(\phi K - \phi E_u) - v_u(E, K, T; \phi) \right] \mathbb{Q}\left(\tau_e \in du \middle| \mathcal{F}_t \right) \\
& + \int_t^T e^{-r(u-t)} \left[(\phi K - \phi E_u) - v_u(E, K, T; \phi) \right] \mathcal{L} \mathbb{Q}\left(\tau_e \in du \middle| \mathcal{F}_t \right) \\
& - e^{-r(t-t)} \left[(\phi K - \phi E_t) - v_t(E, K, T; \phi) \right] \mathbb{Q}\left(\tau_e = t \middle| \mathcal{F}_t \right).
\end{aligned}$$

Because $\mathcal{L} v_t(S, K, T; \phi) = 0$, considering that $\mathbb{Q}\left(\tau_e = t \middle| \mathcal{F}_t \right) = 0$—since Proposition 3 assumes that $\phi S_t > \phi E_t$—and using definition (24), equation (B-1) can be simplified to

(B-2)
$$\begin{aligned}
& \mathcal{L} V_t(S, K, T; \phi) \\
= \; & \int_t^T e^{-r(u-t)} \left[(\phi K - \phi E_u) - v_u(E, K, T; \phi) \right] \left(\frac{\partial}{\partial t} + \mathcal{A} \right) \mathbb{Q}\left(\tau_e \in du \middle| \mathcal{F}_t \right),
\end{aligned}$$

where $\mathcal{A}$ is the infinitesimal generator of S. Since

$$\left(\frac{\partial}{\partial t}+\mathcal{A}\right)\mathbb{Q}\left(\tau_e\in du|\,\mathcal{F}_t\right)=0$$

can be interpreted as a Kolmogorov backward equation, the partial differential equation (23) is obtained. ∎

Appendix C. Proof of Proposition 4

For the perpetual American option, the critical asset price is a time-invariant constant, that is, $E_u = E_\infty, \forall u \in [t, T]$. Hence, the limit of equation (10), as the option's maturity date tends to infinity, is given by

$$\begin{aligned}
&\lim_{T\uparrow\infty} V_t\left(S,K,T;\phi\right)\\
=\;&\lim_{T\uparrow\infty} v_t\left(S,K,T;\phi\right)\\
&+\lim_{T\uparrow\infty}\int_t^T e^{-r(u-t)}\left[\left(\phi K-\phi E_\infty\right)-v_u\left(E_\infty,K,T;\phi\right)\right]\mathbb{Q}\left(\tau_e\in du|\,\mathcal{F}_t\right).
\end{aligned}$$

Furthermore, the fair value of a perpetual European put or call option on a dividend-paying asset is equal to zero and, consequently,

$$\begin{aligned}
\lim_{T\uparrow\infty} V_t\left(S,K,T;\phi\right) &= \left(\phi K-\phi E_\infty\right)\int_t^\infty e^{-r(u-t)}\mathbb{Q}\left(\tau_e\in du|\,\mathcal{F}_t\right)\\
&= \left(\phi K-\phi E_\infty\right)\mathbb{E}_{\mathbb{Q}}\left[e^{-r(\tau_e-t)}1\!\!1_{\{\tau_e<\infty\}}\middle|\,\mathcal{F}_t\right], \qquad \text{(C-1)}
\end{aligned}$$

where τ_e is now the first passage time of the underlying asset price to the constant exercise boundary. Hence, equation (C-1) shows that the proposed characterization of the American option converges to the correct perpetual limit for any Markovian underlying price process.

Under the geometric Brownian motion assumption and for $\phi S_t > \phi E_\infty$, solving the stochastic differential equation (2), for $\sigma(t,S)=\sigma$, and redefining the optimal stopping time τ_e as

$$\begin{aligned}
\tau_e &= \inf\left\{u>t:S_u=E_\infty\right\}\\
&= \inf\left\{u>t:-\frac{\phi}{\sigma}\left(r-q-\frac{\sigma^2}{2}\right)(u-t)-\phi\int_t^u dW_v^{\mathbb{Q}}=\frac{\phi}{\sigma}\ln\left(\frac{S_t}{E_\infty}\right)\right\},
\end{aligned}$$

the dividend-adjusted Merton ((1973), p. 174) solution shown in equation (25) follows after applying Shreve ((2004), Theorem 8.3.2).

Under the CEV model, the expectation contained on the right-hand side of equation (C-1) can easily be computed using, for instance, Davydov and Linetsky ((2001), equations 2 and 38), which yields equations (27) and (32) for $\phi = 1$. For the perpetual American call case ($\phi = -1$), equations (27) and (32) also follow from Davydov and Linetsky ((2001), equations 4 and 37), and Abramowitz and Stegun ((1972), equations 13.1.27 and 13.1.29). ∎

Appendix D. Proof of Proposition 7

Given that the random variable Θ is independent of $\mathbb{F}$, equation (59) can be rewritten in terms of the restricted filtration $\mathbb{F}$ as long as the short-term interest rate is replaced by an intensity-adjusted short-rate:

(D-1)
$$
\begin{aligned}
& EEP^0_{t_0}(S, K, T; \phi) \\
= & \sup_{\tau \in \mathcal{T}} \left\{ \mathbb{E}_{\mathbb{Q}} \left[e^{-\int_{t_0}^{\tau} (r_l + \lambda(l,S))dl} (\phi K - \phi S_\tau)^+ \mathbb{1}_{\{\tau_0 > \tau\}} \mathbb{1}_{\{\tau < T\}} \middle| \mathcal{F}_{t_0} \right] \right. \\
& \left. - e^{-\int_{t_0}^{T} r_l dl} \mathbb{E}_{\mathbb{Q}} \left[e^{-\int_{t_0}^{T} \lambda(l,S)dl} (\phi K - \phi S_T)^+ \mathbb{1}_{\{\tau_0 > T\}} \mathbb{1}_{\{\tau < T\}} \middle| \mathcal{F}_{t_0} \right] \right\}.
\end{aligned}
$$

Moreover, since S_t behaves as a pure diffusion process with respect to the filtration $\mathbb{F}$, then definition (5) can be adopted for the first passage time of the underlying price process through the (continuous) early exercise boundary. Hence,

(D-2)
$$
\begin{aligned}
& EEP^0_{t_0}(S, K, T; \phi) \\
= & \mathbb{E}_{\mathbb{Q}} \left[e^{-\int_{t_0}^{\tau_e} (r_l + \lambda(l,S))dl} (\phi K - \phi S_{\tau_e})^+ \mathbb{1}_{\{\tau_0 > \tau_e\}} \mathbb{1}_{\{\tau_e < T\}} \middle| \mathcal{F}_{t_0} \right] \\
& - e^{-\int_{t_0}^{T} r_l dl} \mathbb{E}_{\mathbb{Q}} \left[e^{-\int_{t_0}^{T} \lambda(l,S)dl} (\phi K - \phi S_T)^+ \mathbb{1}_{\{\tau_0 > T\}} \mathbb{1}_{\{\tau_e < T\}} \middle| \mathcal{F}_{t_0} \right].
\end{aligned}
$$

Concerning the first term on the right-hand side of equation (D-2), the Markovian nature of the underlying price process implies that

$$
\begin{aligned}
& \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\tau_e}(r_l+\lambda(l,S))dl} \left(\phi K-\phi S_{\tau_e}\right)^{+} \mathbb{1}_{\{\tau_0>\tau_e\}}\mathbb{1}_{\{\tau_e<T\}} \middle| \mathcal{F}_{t_0}\right] \\
= & \int_{t_0}^{T} \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{u}(r_l+\lambda(l,S))dl} \left(\phi K-\phi S_{u}\right)^{+} \mathbb{1}_{\{\tau_0>u\}}\mathbb{1}_{\{S_u=E_u\}} \middle| \mathcal{F}_{t_0}\right] \mathbb{Q}\left(\tau_e\in du\right|\left.\mathcal{F}_{t_0}\right) \\
= & \int_{t_0}^{T} e^{-\int_{t_0}^{u} r_l dl} \left(\phi K-\phi E_{u}\right)^{+} \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{u}\lambda(l,S)dl} \mathbb{1}_{\{\tau_0>u\}} \middle| \mathcal{F}_{t_0}\right] \mathbb{Q}\left(\tau_e\in du\right|\left.\mathcal{F}_{t_0}\right) \\
\text{(D-3)} \quad = & \int_{t_0}^{T} e^{-\int_{t_0}^{u} r_l dl} \left(\phi K-\phi E_{u}\right)^{+} SP\left(t_0,u\right) \mathbb{Q}\left(\tau_e\in du\right|\left.\mathcal{F}_{t_0}\right),
\end{aligned}
$$

where the last line follows from identity (61).

Using again the Markov property, and since $\{\tau_0>T\}=\{\inf_{t_0\leq l\leq T}\left(S_l\right)>0\}$, then

$$
\begin{aligned}
& e^{-\int_{t_0}^{T} r_l dl}\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{T}\lambda(l,S)dl} \left(\phi K-\phi S_{T}\right)^{+} \mathbb{1}_{\{\tau_0>T\}}\mathbb{1}_{\{\tau_e<T\}} \middle| \mathcal{F}_{t_0}\right] \\
= & e^{-\int_{t_0}^{T} r_l dl} \int_{t_0}^{T} \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{T}\lambda(l,S)dl} \left(\phi K-\phi S_{T}\right)^{+} \mathbb{1}_{\{\tau_0>T\}}\mathbb{1}_{\{S_u=E_u\}} \middle| \mathcal{F}_{t_0}\right] \mathbb{Q}\left(\tau_e\in du\right|\left.\mathcal{F}_{t_0}\right) \\
= & \int_{t_0}^{T} e^{-\int_{t_0}^{u} r_l dl}\mathbb{E}_{\mathbb{Q}}\left\{ e^{-\int_{u}^{T} r_l dl}\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{u}^{T}\lambda(l,S)dl} \left(\phi K-\phi S_{T}\right)^{+} \mathbb{1}_{\left\{\inf_{u\leq l\leq T}\left(S_l\right)>0\right\}} \middle| S_u=E_u\right]\right. \\
\text{(D-4)} \quad & \left. e^{-\int_{t_0}^{u}\lambda(l,S)dl} \mathbb{1}_{\left\{\inf_{t_0\leq l\leq u}\left(S_l\right)>0\right\}} \middle| \mathcal{F}_{t_0}\right\} \mathbb{Q}\left(\tau_e\in du\right|\left.\mathcal{F}_{t_0}\right).
\end{aligned}
$$

Additionally, equation (58) implies that

$$
\text{(D-5)} \quad e^{-\int_{u}^{T} r_l dl}\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{u}^{T}\lambda(l,S)dl} \left(\phi K-\phi S_{T}\right)^{+} \mathbb{1}_{\left\{\inf_{u\leq l\leq T}\left(S_l\right)>0\right\}} \middle| S_u=E_u\right] = v_u^0\left(E,K,T;\phi\right).
$$

Therefore, equations (61) and (D-5) allow equation (D-4) to be rewritten as

$$
\begin{aligned}
\text{(D-6)} \quad & e^{-\int_{t_0}^{T} r_l dl}\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{T}\lambda(l,S)dl} \left(\phi K-\phi S_{T}\right)^{+} \mathbb{1}_{\{\tau_0>T\}}\mathbb{1}_{\{\tau_e<T\}} \middle| \mathcal{F}_{t_0}\right] \\
= & \int_{t_0}^{T} e^{-\int_{t_0}^{u} r_l dl} v_u^0\left(E,K,T;\phi\right) SP\left(t_0,u\right) \mathbb{Q}\left(\tau_e\in du\right|\left.\mathcal{F}_{t_0}\right).
\end{aligned}
$$

Combining equations (D-3) and (D-6), the pricing solution (60) follows immediately. ∎

Appendix E. Proof of Proposition 8

Using Carr and Linetsky ((2006), Proposition 5.1), then

$$\begin{aligned}\mathbb{Q}\left(S_u \leq E_u \middle| S_l = E_l\right) &= \mathbb{Q}\left\{ e^{\int_l^u \alpha_s ds}\left[\left|\bar{\beta}\right| R^{(v)}_{\tau(l,u)}\right]^{\frac{1}{|\bar{\beta}|}} \leq E_u \middle| R^{(v)}_{\tau(l,l)} = \frac{1}{\left|\bar{\beta}\right|} E_l^{\left|\bar{\beta}\right|}\right\} \\ &= \mathbb{Q}\left\{ \frac{\left[R^{(v)}_{\tau(l,u)}\right]^2}{\tau\left(l,u\right)} \leq \frac{k\left(l,u;E_u\right)^2}{\tau\left(l,u\right)} \middle| R^{(v)}_{\tau(l,l)} = k\left(l,l;E_l\right)\right\}, \qquad \text{(E-1)}\end{aligned}$$

where $\left\{R^{(v)}_{\tau(l,u)}; u \geq l\right\}$ is a Bessel process of index v and started at $R^{(v)}_{\tau(l,l)} = k\left(l,l;E_l\right)$.

For $v \geq 0$, i.e. $c \geq \frac{1}{2}$, Carr and Linetsky ((2006), p. 318) show that the process $\frac{\left[R^{(v)}_{\tau(l,u)}\right]^2}{\tau(l,u)}$ possesses a non-central chi-square distribution with $2\left(1+v\right)$ degrees of freedom and a non-centrality parameter equal to its starting value. Consequently,

$$\mathbb{Q}\left(S_u \leq E_u \middle| S_l = E_l\right) = 1 - Q_{\chi^2\left(2(v+1), \frac{k(l,l;E_l)^2}{\tau(l,u)}\right)}\left(\frac{k\left(l,u;E_u\right)^2}{\tau\left(l,u\right)}\right), \qquad \text{(E-2)}$$

which is equivalent to equation (64) for $v \geq 0$.

If $v < 0$, then Carr and Linetsky ((2006), Proposition 5.3) can be used with $\mu = -v$, and equation (E-1) is rewritten as

$$\begin{aligned}\mathbb{Q}\left(S_u \leq E_u \middle| S_l = E_l\right) &= \mathbb{E}^{(-v)}_{k(l,l;E_l)}\left[\left(\frac{R_{\tau(l,u)}}{k\left(l,l;E_l\right)}\right)^{2v} \mathbb{1}_{\left\{\frac{\left(R_{\tau(l,u)}\right)^2}{\tau(l,u)} \leq \frac{k(l,u;E_u)^2}{\tau(l,u)}\right\}}\right] \\ &= \left[\frac{\tau\left(l,u\right)}{k\left(l,l;E_l\right)^2}\right]^v \mathbb{E}^{(-v)}_{k(l,l;E_l)}\left[\left(\frac{\left(R_{\tau(l,u)}\right)^2}{\tau\left(l,u\right)}\right)^v \mathbb{1}_{\left\{\frac{\left(R_{\tau(l,u)}\right)^2}{\tau(l,u)} \leq \frac{k(l,u;E_u)^2}{\tau(l,u)}\right\}}\right], \qquad \text{(E-3)}\end{aligned}$$

where the expectation is taken with respect to the law of a Bessel process of index $(-v)$ and started at $k\left(l,l;E_l\right)$. Since the process $\frac{\left[R^{(-v)}_{\tau(l,u)}\right]^2}{\tau(l,u)}$ follows a non-central chi-square law with $2\left(1-v\right)$ degrees of freedom and non-centrality parameter $\frac{k(l,l;E_l)^2}{\tau(l,u)}$, then equation (E-3) yields

$$\mathbb{Q}\left(S_u \leq E_u \middle| S_l = E_l\right) = \left[\frac{\tau\left(l,u\right)}{k\left(l,l;E_l\right)^2}\right]^v \Phi^- \left[v, \frac{k\left(l,u;E_u\right)^2}{\tau\left(l,u\right)}; 2\left(1-v\right), \frac{k\left(l,l;E_l\right)^2}{\tau\left(l,u\right)}\right], \qquad \text{(E-4)}$$

which agrees with equation (64) for $v < 0$. ∎

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Table 1: Prices of American Put Options under the Merton (1973) Model, with $S_{t_0} = \$100$ and $T - t_0 = 0.5$ years

			American put									
				Early exercise boundary specification								
Option parameters	Strike	Europ.	Exact	Const.	Exp.	ExpC.	4d Pol.	5d Pol.	CJM	HSY2000	HSY10	EXP3
	80	0.215	0.219	0.218	0.219	0.219	0.219	0.219	0.219	0.219	0.219	0.220
$r = 7\%$	90	1.345	1.386	1.376	1.385	1.386	1.386	1.386	1.386	1.386	1.386	1.387
$q = 3\%$	100	4.578	4.783	4.750	4.778	4.781	4.782	4.782	4.781	4.782	4.782	4.784
$\sigma = 20\%$	110	10.421	11.098	11.049	11.092	11.097	11.097	11.097	11.095	11.097	11.098	11.099
	120	18.302	20.000	20.000	20.000	19.999	19.999	19.999	19.996	20.000	20.002	20.000
	80	2.651	2.689	2.676	2.687	2.688	2.688	2.688	2.688	2.689	2.689	2.690
$r = 7\%$	90	5.622	5.722	5.694	5.719	5.721	5.721	5.721	5.720	5.722	5.722	5.724
$q = 3\%$	100	10.021	10.239	10.190	10.233	10.237	10.237	10.237	10.236	10.239	10.238	10.240
$\sigma = 40\%$	110	15.768	16.181	16.110	16.173	16.180	16.179	16.179	16.177	16.181	16.181	16.183
	120	22.650	23.360	23.271	23.350	23.358	23.358	23.358	23.355	23.359	23.359	23.362
	80	1.006	1.037	1.029	1.036	1.037	1.037	1.037	1.037	1.037	1.037	1.038
$r = 7\%$	90	3.004	3.123	3.098	3.120	3.122	3.122	3.122	3.122	3.123	3.123	3.125
$q = 0\%$	100	6.694	7.035	6.985	7.029	7.034	7.034	7.034	7.032	7.035	7.035	7.037
$\sigma = 30\%$	110	12.166	12.955	12.882	12.946	12.953	12.953	12.953	12.951	12.955	12.953	12.957
	120	19.155	20.717	20.650	20.710	20.716	20.716	20.716	20.713	20.717	20.718	20.719
	80	1.664	1.664	1.664	1.664	1.664	1.664	1.664	1.664	1.664	1.664	1.664
$r = 3\%$	90	4.495	4.495	4.495	4.495	4.495	4.495	4.495	4.495	4.495	4.495	4.495
$q = 7\%$	100	9.251	9.250	9.251	9.251	9.251	9.251	9.251	9.251	9.251	9.251	9.251
$\sigma = 30\%$	110	15.798	15.798	15.798	15.798	15.798	15.798	15.798	15.798	15.798	15.798	15.798
	120	23.706	23.706	23.706	23.706	23.706	23.706	23.706	23.706	23.706	23.706	23.706
Mean Absolute Percentage Error				0.407%	0.054%	0.020%	0.017%	0.015%	0.026%	0.003%	0.005%	0.023%
CPU (seconds)			451.32	0.01	0.03	2.12	8.87	10.07	1.91	3,215.35	0.17	0.08

Table 1 values American put options under the Merton (1973) model and for different specifications of the exercise boundary. The third column contains European put prices, while the exact American put values (fourth column) are based on the binomial model with 15,000 time steps. The fifth, sixth and seventh columns report the American put prices associated with the constant, the exponential, and the exponential-constant boundary specifications, as given by equations (44), (45), and (46). The eighth and ninth columns are both based on a polynomial boundary—see equation (47)—with four and five degrees of freedom, respectively. The American put prices contained in the tenth column are obtained from the exercise boundary of equation (48). The next two columns implement the full (with 2,000 time steps) and the 10-point recursive methods of Huang, Subrahmanyam, and Yu (1996). The last column presents the American put prices generated by the three-point multipiece exponential method of Ju (1998).

Table 2: Prices of American Put Options under the CEV Model, with $\beta = 3$, $S_{t_0} = \$100$ and $T - t_0 = 0.5$ years

			American put								
				Early exercise boundary specification							
Option parameters	Strike	Europ.	Exact	Const.	Exp.	ExpC.	4d Pol.	5d Pol.	CJM	DT	KY
	80	0.159	0.162	0.162	0.162	0.162	0.162	0.162	0.162	0.162	0.162
$r = 7\%$	90	1.255	1.297	1.287	1.296	1.296	1.296	1.296	1.294	1.297	1.297
$q = 3\%$	100	4.579	4.792	4.759	4.787	4.789	4.789	4.790	4.786	4.792	4.791
$\delta = 0.02$	110	10.542	11.215	11.164	11.208	11.213	11.213	11.213	11.209	11.214	11.214
	120	18.452	20.025	20.011	20.021	20.023	20.024	20.024	20.024	20.024	20.027
	80	2.293	2.331	2.319	2.329	2.330	2.330	2.330	2.329	2.331	2.331
$r = 7\%$	90	5.385	5.491	5.463	5.487	5.489	5.489	5.489	5.486	5.491	5.491
$q = 3\%$	100	10.030	10.262	10.212	10.255	10.260	10.260	10.260	10.256	10.261	10.261
$\delta = 0.04$	110	16.043	16.474	16.401	16.464	16.471	16.471	16.471	16.465	16.472	16.472
	120	23.132	23.843	23.751	23.832	23.840	23.840	23.841	23.833	23.839	23.840
	80	0.822	0.852	0.844	0.851	0.851	0.851	0.851	0.850	0.851	0.851
$r = 7\%$	90	2.843	2.969	2.944	2.965	2.967	2.967	2.967	2.964	2.968	2.969
$q = 0\%$	100	6.698	7.060	7.008	7.053	7.058	7.058	7.058	7.053	7.060	7.060
$\delta = 0.03$	110	12.371	13.175	13.099	13.165	13.173	13.172	13.173	13.166	13.174	13.172
	120	19.493	20.992	20.915	20.982	20.989	20.989	20.990	20.983	20.989	20.993
	80	1.419	1.419	1.419	1.419	1.419	1.419	1.419	1.419	1.419	1.419
$r = 3\%$	90	4.311	4.311	4.311	4.311	4.311	4.311	4.311	4.311	4.311	4.311
$q = 7\%$	100	9.254	9.254	9.254	9.254	9.254	9.254	9.254	9.254	9.254	9.254
$\delta = 0.03$	110	15.980	15.980	15.980	15.980	15.980	15.980	15.980	15.980	15.980	15.980
	120	23.978	23.978	23.978	23.978	23.978	23.978	23.978	23.978	23.978	23.978
Mean Absolute Percentage Error				0.384%	0.054%	0.023%	0.022%	0.021%	0.064%	0.006%	0.007%
CPU (seconds)			931.30	7.71	10.40	11.21	30.10	34.14	5.29	5,339.28	0.35

Table 2 values American put options under the CEV model and for different specifications of the exercise boundary. The third column contains European put prices, while the exact American put values (fourth column) are based on the Crank-Nicolson method with 15,000 time intervals and 10,000 space steps. The fifth, sixth and seventh columns report the American put prices associated with the constant, the exponential, and the exponential-constant boundary specifications, as given by equations (44), (45), and (46). The eighth and ninth columns are both based on a polynomial boundary—see equation (47)—with four and five degrees of freedom, respectively. The American put prices contained in the tenth column are obtained from the exercise boundary specification of equation (48). The last two columns implement the full (with 2,000 time steps) and the 10-point recursive methods of Huang, Subrahmanyam, and Yu (1996), as suggested by Detemple and Tian (2002) and Kim and Yu (1996), respectively.

Table 3: Prices of American Call Options under the CEV Model, with $\beta = 1$, $S_{t_0} = \$100$ and $T - t_0 = 1$ year

Option parameters	Strike	Europ.	American call								
				Early exercise boundary specification							
			Exact	Const.	Exp.	ExpC.	2d Pol.	3d Pol.	CJM	DT	KY
	80	23.370	23.370	23.370	23.370	23.370	23.370	23.370	23.370	23.370	23.373
$r = 7\%$	90	15.735	15.735	15.735	15.735	15.735	15.735	15.735	15.735	15.735	15.732
$q = 3\%$	100	9.635	9.635	9.635	9.635	9.635	9.635	9.635	9.635	9.635	9.626
$\delta = 2$	110	5.315	5.315	5.315	5.315	5.315	5.315	5.315	5.315	5.315	5.306
	120	2.630	2.630	2.630	2.630	2.630	2.630	2.630	2.630	2.630	2.625
	80	28.249	28.254	28.253	28.253	28.253	28.253	28.253	28.254	28.254	28.243
$r = 7\%$	90	22.204	22.205	22.205	22.205	22.205	22.205	22.205	22.205	22.205	22.183
$q = 3\%$	100	17.083	17.083	17.083	17.083	17.083	17.083	17.083	17.083	17.083	17.053
$\delta = 4$	110	12.870	12.870	12.870	12.870	12.870	12.870	12.870	12.870	12.870	12.835
	120	9.499	9.499	9.499	9.499	9.499	9.499	9.499	9.499	9.499	9.465
	80	28.022	28.022	28.022	28.022	28.022	28.022	28.022	28.022	28.022	28.023
$r = 7\%$	90	21.061	21.061	21.061	21.061	21.061	21.061	21.061	21.061	21.061	21.051
$q = 0\%$	100	15.221	15.221	15.221	15.221	15.221	15.221	15.221	15.221	15.221	15.201
$\delta = 3$	110	10.567	10.567	10.567	10.567	10.567	10.567	10.567	10.567	10.567	10.542
	120	7.047	7.047	7.047	7.047	7.047	7.047	7.047	7.047	7.047	7.024
	80	20.301	21.882	21.794	21.871	21.801	21.879	21.879	21.878	21.881	21.871
$r = 3\%$	90	14.257	15.187	15.100	15.175	15.105	15.184	15.184	15.184	15.186	15.164
$q = 7\%$	100	9.552	10.084	10.015	10.074	10.018	10.081	10.081	10.082	10.082	10.058
$\delta = 3$	110	6.106	6.401	6.353	6.394	6.354	6.398	6.398	6.400	6.400	6.380
	120	3.728	3.886	3.855	3.881	3.856	3.884	3.884	3.885	3.885	3.875
Mean Absolute Percentage Error				0.161%	0.024%	0.154%	0.009%	0.009%	0.005%	0.005%	0.162%
CPU (seconds)			912.31	5.60	7.07	5.60	9.94	10.84	3.87	5,690.36	0.49

Table 3 values American call options under the CEV model and for different specifications of the exercise boundary. The third column contains European call prices, while the exact American call values (fourth column) are based on the Crank-Nicolson method with 15,000 time intervals and 10,000 space steps. The fifth, sixth and seventh columns report the American call prices associated with the constant, the exponential, and the exponential-constant boundary specifications, as given by equations (44), (45), and (46). The eighth and ninth columns are both based on a polynomial boundary—equation (47)—with two and three degrees of freedom, respectively. The American call prices contained in the tenth column are obtained from the exercise boundary specification of equation (48). The last two columns implement the full (with 2,000 time steps) and the 10-point recursive methods of Huang, Subrahmanyam, and Yu (1996), as suggested by Detemple and Tian (2002) and Kim and Yu (1996), respectively.

Table 4: Accuracy of the Polynomial Specification for a Large Sample of Randomly Generated American Puts

	Polynomial specifications					KY
	2nd degree	3rd degree	4th degree	5th degree	6th degree	
Percentage Errors						
mean	-0.0291%	-0.0226%	-0.0200%	-0.0179%	-0.0147%	0.0068%
maximum	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%	0.3953%
minimum	-0.1691%	-0.1162%	-0.1071%	-0.1003%	-0.0902%	-0.0563%
99th percentile	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%	0.1898%
1st percentile	-0.1308%	-0.1037%	-0.0949%	-0.0889%	-0.0758%	-0.0480%
Absolute Percentage Errors						
mean	0.0291%	0.0226%	0.0200%	0.0179%	0.0147%	0.0158%
maximum	0.1691%	0.1162%	0.1071%	0.1003%	0.0902%	0.3953%
minimum	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%
99th percentile	0.1308%	0.1037%	0.0949%	0.0889%	0.0758%	0.1898%

Table 4 reports the pricing errors associated with the valuation of 1,250 randomly generated American put options, under the CEV model and through different polynomial parameterizations of the exercise boundary, as given by equation (47). For comparison purposes, the last column contains the pricing errors associated with the 10-point recursive scheme of Huang, Subrahmanyam, and Yu (1996), as suggested by Kim and Yu (1996). The strike price is always set at \$100 while the other option features were generated from uniform distributions and within the following intervals: instantaneous volatility between 10% and 60%; interest rate and dividend yield between 0% and 10%; underlying spot price between \$70 an \$130; beta between 0 and 4.0; and time-to-maturity ranging from 0 to 3.0 years. The pricing errors produced by the alternative boundary specifications were computed against the Crank-Nicolson method with 15,000 time intervals and 10,000 space steps.

Table 5: American Put Options under the JDCEV Model, with $\bar{\beta} = -0.5$, $c = 0.5$, $b = 0.02$, $S_{t_0} = \$100$ and $T - t_0 = 5$ years

Option parameters	Strike	European put $v_{t_0}^0$	$v_{t_0}^D$	v_{t_0}	American put $EEP_{t_0}^0$	$V_{t_0}^0$	$V_{t_0}^D$	V_{t_0}
	80	1.095	9.772	10.866	1.005	2.100	11.602	13.702
$r = 7\%$	90	1.842	10.993	12.835	2.098	3.940	13.053	16.992
$q = 3\%$	100	2.895	12.215	15.110	4.129	7.024	14.503	21.527
$a = 2$	110	4.302	13.436	17.739	7.697	11.999	15.953	27.953
	120	6.098	14.658	20.756	13.425	19.523	17.404	36.927
	80	1.797	22.017	23.814	5.599	7.397	26.818	34.215
$r = 7\%$	90	2.504	24.769	27.273	7.853	10.358	30.170	40.528
$q = 3\%$	100	3.358	27.521	30.879	10.757	14.115	33.523	47.638
$a = 4$	110	4.363	30.273	34.636	14.430	18.793	36.875	55.668
	120	5.525	33.025	38.550	18.973	24.498	40.227	64.726
	80	1.400	14.570	15.970	2.590	3.990	17.688	21.678
$r = 7\%$	90	2.054	16.391	18.445	4.247	6.301	19.899	26.200
$q = 0\%$	100	2.884	18.213	21.096	6.752	9.636	22.110	31.746
$a = 3$	110	3.906	20.034	23.940	10.425	14.331	24.321	38.652
	120	5.134	21.855	26.989	15.504	20.637	26.532	47.170
	80	5.385	21.080	26.465	3.857	9.242	22.785	32.027
$r = 3\%$	90	7.496	23.715	31.212	5.340	12.836	25.633	38.469
$q = 7\%$	100	10.001	26.351	36.352	7.172	17.173	28.481	45.654
$a = 3$	110	12.891	28.986	41.877	9.392	22.283	31.329	53.612
	120	16.148	31.621	47.768	12.045	28.193	34.177	62.370

Table 5 values American put options under a time-homogeneous JDCEV model, and for the parameter values considered in Table 3. The third column contains European put prices, conditional on no default, as given by equation (58). The fourth column reports the recovery component of the European put contract, which is computed from Carr and Linetsky ((2006), eq. 3.11), and the fifth column yields the sum of the previous two components. The early exercise premium (sixth column) is computed through Propositions 7 and 8, using a polynomial specification for the exercise boundary with four degrees of freedom. The seventh, eighth and ninth columns are given by equations (57), (56) and (54), respectively.