A General Equilibrium Framework for the Affine Class of Term Structure Models

by

João Pedro Vidal Nunes*

CEMAF/ISCTE

Complexo INDEG/ISCTE, Av. Prof. Aníbal Bettencourt 1600-189 Lisboa, Portugal.

Tel: +351 21 7958607. Fax: +351 21 7958605.

E-mail: joao.nunes@iscte.pt.

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Brief Title: **A General Equilibrium Affine Term Structure Model**

Abstract

The Duffie and Kan (1996) model, which can be considered as the most general affine term structure model, was originally specified in terms of risk-adjusted stochastic processes for its state variables. Here, the Duffie and Kan (1996) model is initially fitted into a general equilibrium framework under the physical probability measure, and then its equilibrium specification under an equivalent martingale measure is derived consistently with Duffie and Kan (1996). The resulting analytical solution for the vector of factor' risk premiums enables the econometric estimation of the model' parameters using a "time-series" or a "panel-data" approach, and nests, as special cases, several other specifications already proposed in the literature.

The Duffie and Kan (1996) model of the term structure of interest rates possesses, at least, three appealing features. First, this model is constructed under realistic assumptions, since it incorporates mean reversion, and accommodates both deterministic (Gaussian models) and stochastic volatility. Second, the Duffie and Kan (1996) model is an extremely general framework because it embodies as special cases a large number of well known models previously presented in the literature, such as Vasicek (1977), Langetieg (1980), Cox, Ingersoll, and Ross (1985b), Longstaff and Schwartz (1992a), Fong and Vasicek (1991b), or Chen and Scott (1995a). In fact, the Duffie and Kan (1996) specification can be considered as the most general affine[1] and time-homogeneous multifactor term structure model. Finally, the Duffie and Kan (1996) model is also numerically tractable because it generates an exponential-affine formula for $P(t,T)$, that is for the time-t price of a default-free (and unit face value) pure discount bond expiring at time T (although, under the most general stochastic volatility case, the time-dependent functions involved in such valuation formula can only be recovered through the numerical solution of a system of Riccati differential equations).

Duffie and Kan (1996) start by considering a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, and assume that under the physical probability measure $\mathcal{P}$ the vector of state variables $\underline{X}(t)$ satisfies a stochastic differential equation (SDE) of the generic form

$$d\underline{X}(t) = \underline{v}[\underline{X}(t)]\,dt + \sigma[\underline{X}(t)] \cdot d\underline{W^{\mathcal{P}}}(t), \tag{1}$$

where $\underline{v}[\underline{X}(t)] \in \Re^n$ and $\sigma[\underline{X}(t)] \in \Re^{n\times n}$ satisfy the *Lipschitz* and *growth* conditions required for a unique solution to exist for Equation (1),[2] while $\underline{W^{\mathcal{P}}}(t) \in \Re^n$ is a standard Brownian motion under $\mathcal{P}$ generating the augmented filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$. As usual, the filtration $(\mathcal{F}_t)_{t\geq 0}$ is assumed to be right-continuous with $\mathcal{F}_0$ containing all the $\mathcal{P}-$null sets of $\mathcal{F}$. Then, they argue that it is always possible to derive a probability martingale measure $\mathcal{Q}$ equivalent to $\mathcal{P}$ (that is mutually absolutely continuous) and a standard $\mathcal{Q}$-measured Brownian motion $\underline{W^{\mathcal{Q}}}(t) \in \Re^n$ (with the same standard filtration as $\underline{W^{\mathcal{P}}}(t)$), such that

$$d\underline{X}(t) = \underline{\mu}[\underline{X}(t)]\,dt + \sigma[\underline{X}(t)] \cdot d\underline{W^{\mathcal{Q}}}(t) \tag{2}$$

and where $\underline{\mu}[\underline{X}(t)] \in \Re^n$ is a compatible function of $\underline{v}[\underline{X}(t)]$, $\sigma[\underline{X}(t)]$ and $P(t,T)$, in the sense that this change of drift guarantees the absence of arbitrage opportunities[3] and also preserves an exponential-affine specification for pure discount bond prices. Finally, Duffie and Kan (1996) define what they call a $\left(P, \underline{\mu}, \sigma\right)$ compatible term structure model by specifying an exponential-affine form for $P(t,T)$ and affine formulae for both $\underline{\mu}\left[\underline{X}(t)\right]$ and $\sigma\left[\underline{X}(t)\right]\cdot\sigma\left[\underline{X}(t)\right]'$, where $\sigma\left[\underline{X}(t)\right]'$ denotes the transpose of $\sigma\left[\underline{X}(t)\right]$. In other words, the Duffie and Kan (1996) model was originally defined not under objective probabilities but in terms of risk-adjusted stochastic processes for its state variables, i.e. with respect to a martingale measure $\mathcal{Q}$, which can be understood as the probability measure obtained when a "money market account" is taken as the numeraire of the stochastic intertemporal economy underlying the model under analysis.

The goal of the present article is to derive a Duffie and Kan (1996) model' specification under the physical probability measure $\mathcal{P}$ that is compatible with the formulation given by the authors under the equivalent martingale measure $\mathcal{Q}$. This task can become useful for empirical purposes, namely for the econometric estimation of the Duffie and Kan (1996) model' parameters from a time-series of state variables' values or from a panel-data of market observables (e.g. bond prices), through Kalman filtering techniques. In fact, these parameters can also be estimated from a cross-section of bond prices by using the risk-adjusted processes for the state variables (that is through the best fit between market bond prices and those generated by the model), since assuming that there are no arbitrage opportunities in the bond market is equivalent to say that such interest rate contingent claims can be priced under an equivalent martingale measure $\mathcal{Q}$. However, this latter methodology should be less adequate than the time-series or panel-data approaches, because the model' parameters are assumed to be time-independent. In summary, if the Duffie and Kan (1996) model' parameters are to be estimated through a time-series or a panel-data methodology, the knowledge of the model' specification under the objective probability measure $\mathcal{P}$ is then required, and thus justifies the purpose of this paper. As Duffie and Kan (1994, page 578) notice: "For many applications, it will also be useful to model the distribution of processes under the original probability measure $\mathcal{P}$. Conversion from $\mathcal{P}$ to $\mathcal{Q}$ and back will not be dealt with here, but is an important issue, particularly from the point of view of statistical fitting of the models as well as the measurement of risk."

In order to derive the Duffie and Kan (1996) model' specification under the probability measure $\mathcal{P}$, it will be necessary to fit the model into a general equilibrium framework. This is so, because, from Girsanov's Theorem, the two model specifications (under probability measures $\mathcal{P}$ and $\mathcal{Q}$) are only compatible if $\underline{\mu}\left[\underline{X}(t)\right]$ and $\underline{W^{\mathcal{Q}}}(t)$ are such that:

$$\underline{\mu}\left[\underline{X}(t)\right] = \underline{v}\left[\underline{X}(t)\right] - \sigma\left[\underline{X}(t)\right]\cdot\underline{\Lambda}\left[\underline{X}(t)\right]$$

and

$$d\underline{W^{\mathcal{Q}}}(t) = \underline{\Lambda}\left[\underline{X}(t)\right] dt + d\underline{W^{\mathcal{P}}}(t),$$

where $\underline{\Lambda}\left[\underline{X}(t)\right] \in \Re^n$, which can be interpreted as the time-t vector of market prices of interest rate risk, satisfies the Novikov's condition

$$E_{\mathcal{P}}\left\{\exp\left[\frac{1}{2}\int_0^t \underline{\Lambda}\left(\underline{X}(s)\right)' \cdot \underline{\Lambda}\left(\underline{X}(s)\right) ds\right] \Bigg| \mathcal{F}_0\right\} < \infty,$$

and therefore the Radon-Nikodym derivative

$$\left.\frac{d\mathcal{Q}}{d\mathcal{P}}\right| \mathcal{F}_t \equiv \exp\left\{-\int_0^t \underline{\Lambda}\left[\underline{X}(s)\right]' \cdot d\underline{W^{\mathcal{P}}}(s) - \frac{1}{2}\int_0^t \underline{\Lambda}\left[\underline{X}(s)\right]' \cdot \underline{\Lambda}\left[\underline{X}(s)\right] ds\right\}$$

is a martingale.

Hence, to go from the Duffie and Kan (1996) model specification under the original probability measure $\mathcal{P}$ -hereafter labelled as the $(P, \underline{v}, \underline{\Lambda}, \sigma)$ model- to the $\left(P, \underline{\mu}, \sigma\right)$ equivalent specification, or all the way around, it is necessary to define $\underline{\Lambda}\left[\underline{X}(t)\right]$ explicitly. For that purpose, the Duffie and Kan (1996) model will have to be fitted into a general equilibrium framework, where both the short-term interest rate and the vector of market prices of risk will be endogenously determined in the context of the underlying economy. And, unlike the majority of the general equilibrium term structure models found in the literature, the role of money is going to be explicitly considered, leading to a general equilibrium Duffie and Kan (1996) model of the term structure of *nominal* interest rates.

Next sections are organized as follows. Section 1 presents a brief summary of the Duffie and Kan (1996) model in its known risk-neutral specification. Section 2 states all the assumptions that are required to fit the Duffie and Kan (1996) model into a general equilibrium framework. In sections 3, 4 and 5, general formulae for the equilibrium short-term interest rate and for the equilibrium factor risk premiums are derived, always in *nominal* terms: first, within the context of a production economy; then, under a consumption-based CAPM; and finally, assuming a pure exchange economy. In Section 6, a general equilibrium Duffie and Kan (1996) model is derived under a constant relative risk aversion economy (both with power and log utility functions). Finally, Section 7 summarizes the conclusions. All accessory results are relegated to the Appendices.

1 Duffie and Kan (1996) model: a summary

The Duffie and Kan (1996) model imposes an exponential-affine form for the time-t price of a riskless (unit face value) pure discount bond, that is

$$P(t,T) = \exp\left[A(\tau) + \underline{B}'(\tau) \cdot \underline{X}(t)\right], \tag{3}$$

where $\tau = T - t$ is the time-to-maturity of the zero-coupon bond, $\cdot$ denotes the inner product in $\Re^n$, and $\underline{X}(t) \in \Re^n$ is the time-t vector of state variables. In order to respect the boundary condition $P(T,T) = 1$, the time-homogeneous functions $A(\tau) \in \Re$ and $\underline{B}(\tau) \in \Re^n$ must be such that $A(0) = 0$ and $\underline{B}(0) = \underline{0}$. Moreover, the function $P(t,T)$ is assumed to be continuously differentiable in the time-to-maturity and twice continuously differentiable in the state-vector.

Alternatively to zero-coupon bond prices, the model can be equivalently specified in terms of the riskless instantaneous spot interest rate. Because $A(\cdot)$ and $\underline{B}(\cdot)$ are continuously differentiable (since it is assumed that $P(t,T) \equiv P(\underline{X}(t);\tau) \in C^{2,1}(\mathbf{D} \times [0,\infty[)$, where $\mathbf{D} \subseteq \Re^n$ represents the admissible domain of the model' state variables), it follows from Equation (3) that the time-t short-term interest rate $r(t)$ is an affine function of the n factors:

$$\begin{aligned} r(t) &= \lim_{\tau \to 0}\left[-\frac{\ln P(t,T)}{\tau}\right] \\ &= f + \underline{G}' \cdot \underline{X}(t), \end{aligned} \tag{4}$$

where $f = -\left.\frac{\partial A(\tau)}{\partial \tau}\right|_{\tau=0}$, and the i^{th} element of vector $\underline{G} \in \Re^n$ is defined as $g_i = -\left.\frac{\partial B_i(\tau)}{\partial \tau}\right|_{\tau=0}$, being $B_i(\tau)$ the i^{th} element of vector $\underline{B}(\tau)$.

Concerning the dynamics of the model' factors, Duffie and Kan (1996) assume that the n state variables follow, under a martingale measure $\mathcal{Q}$, a parametric Markov diffusion process, where the drift and the variance of these risk-adjusted stochastic processes also have an affine form, to support[4] the exponential-affine specification of Equation (3):

$$d\underline{X}(t) = \left[a \cdot \underline{X}(t) + \underline{b}\right] dt + \Sigma \cdot \sqrt{V^D(t)} \cdot d\underline{W}^{\mathcal{Q}}(t), \underline{X}(t) \in \mathbf{D}, \tag{5}$$

where $a, \Sigma \in \Re^{n \times n}$, $\underline{b} \in \Re^n$,

$$\sqrt{V^D(t)} \equiv diag\left\{\sqrt{v_1(t)}, \ldots, \sqrt{v_n(t)}\right\},$$

$$v_i(t) \equiv \alpha_i + \underline{\beta_i}' \cdot \underline{X}(t), \quad for\ i = 1, ..., n,$$

$\alpha_i \in \Re$, $\underline{\beta_i} \in \Re^n$, $\underline{W}^{\mathcal{Q}}(t) \in \Re^n$ is a vector of n independent Brownian motions under measure $\mathcal{Q}$, and

$$\mathbf{D} = \left\{ \underline{X} \in \Re^n : \alpha_i + \underline{\beta_i}' \cdot \underline{X} \geq 0, i = 1, \ldots, n \right\} \tag{6}$$

is the admissible domain of the model' state variables. Notice that this model specification incorporates mean reversion ($\underline{X}(t)$ mean reverts towards $a^{-1} \cdot \underline{b}$, as long as matrix a is negative definite), and accommodates both deterministic (if $\underline{\beta_i} = \underline{0}, \forall i$) or stochastic volatility (if $\exists i : \underline{\beta_i} \neq \underline{0}$) formulations. Hereafter, condition A of Duffie and Kan (1996, page 387) will be always assumed, which ensures that a unique (strong) solution $\underline{X}(t) \in \mathbf{D}$ exists for the SDE (5).

Equations (3) -or (4)- and (5) summarize the most general stochastic volatility specification of the Duffie and Kan (1996) model (since $\underline{\beta_i}$ is not constrained to be equal to $\underline{0}$, for all i). Applying Itô's lemma, it follows that, under this general specification, the time-t price, $Y[\underline{X}(t), t] \in C^{2,1}(\mathbf{D} \times [0, \infty[)$, of an interest rate contingent claim, with a continuous "dividend yield" $i[\underline{X}(t), t]$, must satisfy the following fundamental parabolic partial differential equation (PDE), subject to the appropriate boundary conditions:

$$\mathcal{D}Y(\underline{x}, t) + \frac{\partial Y(\underline{x}, t)}{\partial t} - r(t) Y(\underline{x}, t) = -i(\underline{x}, t), \underline{x} \in \mathbf{D}, \tag{7}$$

being $\mathcal{D}$ the second-order differential operator[5]

$$\mathcal{D}Y(\underline{x}, t) \equiv \frac{\partial Y(\underline{x}, t)}{\partial \underline{x}'} \cdot [a \cdot \underline{x} + \underline{b}] + \frac{1}{2} tr \left[\frac{\partial^2 Y(\underline{x}, t)}{\partial \underline{x} \partial \underline{x}'} \cdot \Sigma \cdot V^D(t) \cdot \Sigma' \right],$$

with $V^D(t) \equiv diag\{v_1(t), \ldots, v_n(t)\}$, and where the function $tr(\cdot)$ returns the trace of a square matrix. However, and as Duffie and Kan (1994) point out, PDE (7) can only be solved, for path-independent interest rate contingent claims, by a finite-difference method or, for large n, by Monte Carlo simulation. The only exception seems to be the valuation of default-free pure discount bonds, for which an exact quasi-closed form solution is provided by Duffie and Kan (1996). Using Equations (3.9) and (3.10) of Duffie and Kan (1996),[6] first the *duration* vector $\underline{B}'(\tau)$ must be found through the solution of a system of n Riccati differential equations (for instance, by using a fifth order Runge-Kutta method),

$$\frac{\partial}{\partial \tau} \underline{B}'(\tau) = -\underline{G}' + \underline{B}'(\tau) \cdot a + \frac{1}{2} \sum_{k=1}^{n} \left[\sum_{j=1}^{n} B_j(\tau) \varepsilon_{jk} \right]^2 \underline{\beta_k}', \tag{8}$$

subject to the initial condition $\underline{B}(0) = \underline{0}$, and where ε_{jk} is the j^{th}-row k^{th}-column element of matrix Σ. Then, $A(\tau)$ is obtained through the solution of a first order ordinary differential equation (for instance, by using Romberg's integration method),

$$\frac{\partial}{\partial \tau} A(\tau) = -f + \underline{B}'(\tau) \cdot \underline{b} + \frac{1}{2} \sum_{k=1}^{n} \left[\sum_{j=1}^{n} B_j(\tau) \varepsilon_{jk} \right]^2 \alpha_k, \tag{9}$$

subject to the initial condition $A(0) = 0$. Finally, $P(t,T)$ is given by Equation (3). However, under this general specification of the Duffie and Kan (1996) model, even the above ODEs must be solved numerically. Recently, Duffie, Pan, and Singleton (1998) proposed exact Fourier transform pricing solutions for an affine jump-diffusion model that nests, as a special case, the Duffie and Kan (1996) framework under analysis, and Nunes, Clewlow, and Hodges (1999), using an Arrow-Debreu pricing approach, derived fast and accurate valuation formulae for European-style interest rate contingent claims under the general stochastic volatility class of affine term structure models.

The main advantage of the Duffie and Kan (1996) framework is its generality: all time-homogeneous exponential-affine models presented in the literature can be easily nested into the specification given by Equations (4) and (5), through self-evident parameters' restrictions (Table 1 illustrates some examples). Therefore, the general equilibrium setup that will be constructed in the present paper is also applicable to any of such models.

2 General equilibrium assumptions

The following assumptions represent a synthesis between the consumption-based CAPM of Breeden (1979), the continuous-time pure exchange economy of Lucas (1978), and the cash-in-advance one-country economy of Lucas (1982), while the notation is intended to follow that used by Cox, Ingersoll, and Ross (1985a):

A.1) There is a single physical good, which can be allocated to consumption or investment.

A.2) The stochastic intertemporal one-country economy that will be considered possess a finite time horizon $\mathcal{T} = [0, T]$. Uncertainty is represented by a complete filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathcal{T}}, \mathcal{P}\right)$, where all the information accruing to all the agents in the economy is described by a filtration $(\mathcal{F}_t)_{t \in \mathcal{T}}$ satisfying the usual conditions: namely, $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. The vector $\underline{W^{\mathcal{P}}}(t) \in \Re^n$ will represent a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$, and $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ will denote the $\mathcal{P}-$augmentation of the natural filtration generated by $\underline{W^{\mathcal{P}}}(t)$.

A.3) There are n state variables that determine the general state of the economy (both in real and monetary terms) through the following stochastic process, and under the probability measure $\mathcal{P}$:

$$d\underline{X}(t) = \left[\bar{a} \cdot \underline{X}(t) + \underline{\bar{b}}\right] dt + \Sigma \cdot \sqrt{V^D(t)} \cdot d\underline{W}^{\mathcal{P}}(t), \tag{10}$$

where $\bar{a} \in \Re^{n \times n}$, $\underline{\bar{b}} \in \Re^n$, and $d\underline{W}^{\mathcal{P}}(t) \in \Re^n$ is a vector of n independent Brownian increments under the objective probability measure. Hence, $\underline{\nu}[\underline{X}(t)] = \bar{a} \cdot \underline{X}(t) + \underline{\bar{b}}$ and $\sigma[\underline{X}(t)] = \Sigma \cdot \sqrt{V^D(t)}$. This stochastic differential equation is intended to represent the non-risk-adjusted stochastic process followed by the state variables of the Duffie and Kan (1996) model. Thus, the diffusion is the same as in Equation (5), and the drift was defined as another affine function of the n factors (ensuring consistency with the exponential-affine form for pure discount bond prices). The goal is precisely to determine a consistent relation between a and $\bar{a}$ as well as between $\underline{b}$ and $\underline{\bar{b}}$.

A.4) There exist m distinct production processes (or production firms) that define m investment opportunities in the economy, whose dynamics are modelled through the following SDE:

$$d\underline{S}(t) = I_S(t) \cdot \underline{\mu_S}(q, M, \underline{S}, \underline{X}, t)\, dt + I_S(t) \cdot E(q, M, \underline{S}, \underline{X}, t) \cdot d\underline{W}^{\mathcal{P}}(t). \tag{11}$$

The i^{th} element of $\underline{S}(t) \in \Re^m$, denoted by $S_i(t)$, represents the *nominal* value of the i^{th} production firm at time t, $I_S(t) = diag\{S_1(t), \ldots, S_m(t)\}$ and therefore the production processes have stochastically constant returns to scale, $\underline{\mu_S}(q, M, \underline{S}, \underline{X}, t) \in \Re^m$ is the vector of expected rates of return on the production activities, $E(q, M, \underline{S}, \underline{X}, t) \in \Re^{m \times n}$ is assumed to be such that $E(q, M, \underline{S}, \underline{X}, t) \cdot E(q, M, \underline{S}, \underline{X}, t)'$ is positive definite, $q(t)$ denotes the time-t aggregate output of the economy, and $M(t)$ represents the time-t money supply level. Each firm's value is represented by just one (perfectly divisible) share, i.e. $S_i(t)$ can be though of as being the value of the i^{th} production firm share.

A.5) The real aggregate production output is exogenously determined by the following diffusion process:

$$\frac{dq(t)}{q(t)} = \mu_q(q, \underline{X}, t)\, dt + \underline{\sigma_q}(q, \underline{X}, t)' \cdot d\underline{W}^{\mathcal{P}}(t), \tag{12}$$

where $\mu_q(q, \underline{X}, t) \in \Re$ is the time-t expected rate of change in the aggregate output, and $\underline{\sigma_q}(q, \underline{X}, t) \in \Re^n$ is the vector of volatilities for the rate of change in the aggregate output. Assumption A.5 corresponds to the main difference between the pure exchange economy considered here and the Cox, Ingersoll, and Ross (1985a) type of production economy.

A.6) The money supply is exogenously determined by the following diffusion process:

$$\frac{dM(t)}{M(t)} = \mu_M(M, \underline{X}, t)\, dt + \underline{\sigma_M}(M, \underline{X}, t)' \cdot d\underline{W}^{\mathcal{P}}(t), \tag{13}$$

where $\mu_M(M, \underline{X}, t) \in \Re$ is the time-t expected growth rate of money supply, and $\underline{\sigma_M}(M, \underline{X}, t) \in \Re^n$ is the vector of volatilities for the money supply growth rate.

A.7) There are $(n-m)$ infinitely divisible financial contingent claims, whose net supply is zero, and whose *nominal* value evolves accordingly to the following stochastic process:

$$d\underline{F}(t) = I_F(t) \cdot \underline{\mu_F}(q, M, \underline{S}, \underline{X}, t)\, dt + I_F(t) \cdot H(q, M, \underline{S}, \underline{X}, t) \cdot d\underline{W}^{\mathcal{P}}(t), \tag{14}$$

where the i^{th} element of $\underline{F}(t) \in \Re^{n-m}$, denoted by $F_i(t)$, represents the time-t price of the i^{th} contingent claim, $I_F(t) = diag\{F_1(t), \ldots, F_{n-m}(t)\}$, $\underline{\mu_F}(q, M, \underline{S}, \underline{X}, t) \in \Re^{n-m}$ is the vector of expected rates of return (dividend-inclusive) on the $(n-m)$ financial contingent claims, and $H(q, M, \underline{S}, \underline{X}, t) \in \Re^{(n-m)\times n}$.

A.8) There are no taxes or transaction costs, and all trades take place at equilibrium prices.

A.9) There exists a market for instantaneous borrowing and lending at a *nominal* risk-free interest rate of $r(t)$.

A.10) There exists a fixed number of individuals, all identical in terms of their endowments and preferences, and all having homogeneous probability beliefs about future states of the world. Thus, it can be automatically assumed that markets are dynamically complete, because as said in Cox, Ingersoll, and Ross (1981, page 779): "For an economy of identical investors, prices will be set as if markets were complete, regardless of their actual scope". Moreover, each individual seeks to maximize the expected value of a time-additive and state-independent von Neumann-Morgenstern utility function for lifetime consumption, that is wishes to maximize the quantity

$$E_t \left\{ \int_t^T u\left[C\left(s\right), s\right] ds \middle| V\left(t\right) = v = \sum_{i=1}^{m} S_i\left(t\right) \ and \ \underline{X}\left(t\right) = \underline{x} \right\},$$

where t denotes the current time, T represents the terminal date, the expectation is conditional on $\mathcal{F}_t$ and computed under measure $\mathcal{P}$, $u\left[\cdot\right]$ is a von Neumann-Morgenstern period utility function, $C\left(s\right)$ represents the amount of the single physical good consumed at time s, and $\underline{x}$ denotes the current state of the economy. $V\left(t\right)$ is the time-t (i.e. current) pre-decision *nominal* wealth, since it is being assumed that the initial endowment of the representative agent corresponds to one share of each production firm.

A.11) The unit-velocity version of the *Quantity Theory of Money* will be assumed hereafter, that is

$$\frac{M\left(t\right)}{p\left(t\right) q\left(t\right)} = 1, \tag{15}$$

where $p\left(t\right)$ is the time-t price level for the single physical good. Such working hypothesis is just a consequence of the following three underlying assumptions:

A.11.1) In the economy under analysis all agents are subject to a cash-in-advance constraint (also known as the *Clower constraint*), in the sense that all goods can be purchased only with currency accumulated in advance, i.e.

$$N\left(t\right) = p\left(t\right) C\left(t\right), \tag{16}$$

where $N\left(t\right)$ is the time-t demand for money. This constraint justifies the existence of money in the economy, because as argued by Lucas (1982, page 342): "...agents will hold non-interest-bearing units of that currency in exactly the amount needed to cover their perfectly predictable current-period goods purchases". Instead, one could have considered, for instance, the existence of real cash balances in the direct utility function, while assuming that $q\left(t\right)$ and $M\left(t\right)$ were the only state variables, as done by Bakshi and Chen (1996). Although such procedure would be more realistic, it would also create two problems: first, the choice of state variables would not be consistent with the Duffie and Kan (1996) model specification under analysis; second, the derivation of a closed-form expression for $\underline{\Lambda}\left[\underline{X}\left(t\right)\right]$ would require the use of a log utility function, restricting the type of preferences under consideration.

A.11.2) In equilibrium, the money supply equals the demand for money:

$$M(t) = N(t). \tag{17}$$

A.11.3) In the pure exchange economy under analysis, all output is consumed:

$$C(t) = q(t). \tag{18}$$

Combining Equations (16), (17), and (18), Equation (15) follows immediately.

Initially, assumption A.5 will be ignored, i.e. a Cox, Ingersoll, and Ross (1985a) type of production economy will be considered, but the results obtained are going to depend on the indirect utility function. Then, this paper will move towards the consumption-based CAPM of Breeden (1979), obtaining results that depend on the direct utility function but are still related to the endogenous consumption process. Finally and following Bakshi and Chen (1997b), assumption A.5 will be imposed, a pure exchange economy will be completely identified, and all the relevant results will be stated in terms of the utility of consumption and as a function of the exogenous output and money supply processes (therefore avoiding the need to solve the Hamilton-Jacobi-Bellman equation for the utility of wealth or for the endogenous consumption process). Consequently, it will be possible to consider a general equilibrium framework based on preference assumptions more realistic than those implied by the usual log utility of consumption. Moreover, since a monetary economy is considered, the general equilibrium Duffie and Kan (1996) model specification that will emerge is a term structure model of *nominal* interest rates.

3 Portfolio selection problem

3.1 The budget constraint

The representative agent in the economy can choose amongst three different types of investment opportunities: *i*) To trade the equity shares issued by the m production firms; *ii*) To trade $(m-n)$ financial contingent claims; and *iii*) To buy or sell instantaneous *nominal* risk-free zero-coupon bonds.

Hence, the representative agent must observe the following budget constraint, where, for clarity, all functional dependencies, except time-dependencies, will be suppressed:

$$dV(t) = V(t)\,\underline{\omega_S}(t)' \cdot I_S^{-1}(t) \cdot d\underline{S}(t) + V(t)\,\underline{\omega_F}(t)' \cdot I_F^{-1}(t) \cdot d\underline{F}(t) \tag{19}$$

$$+V(t)\left[1-\underline{\omega_S}(t)'\cdot\underline{1}-\underline{\omega_F}(t)'\cdot\underline{1}\right]r(t)\,dt-p(t)\,C(t)\,dt,$$

where $\underline{\omega_S}(t) \in \Re^m$, its i^{th} element, $\omega_{S_i}(t)$, is the proportion of the current wealth invested in the i^{th} production firm, $\underline{\omega_F}(t) \in \Re^{n-m}$, its i^{th} element, $\omega_{F_i}(t)$, is the proportion of the current wealth invested in the i^{th} financial contingent claim, and $r(t)$ is the instantaneous *nominal* risk-free time-t interest rate. Considering Equations (11) and (14), the above stochastic differential equation can be restated as:

$$\begin{aligned} dV(t) &= \left\{\underline{\omega_S}(t)'\cdot\left[\underline{\mu_S}(t)-r(t)\underline{1}\right]V(t)+\underline{\omega_F}(t)'\cdot\left[\underline{\mu_F}(t)-r(t)\underline{1}\right]V(t)\right. \\ &\quad \left.+V(t)\,r(t)-p(t)\,C(t)\right\}dt \\ &\quad +V(t)\left[\underline{\omega_S}(t)'\cdot E(t)+\underline{\omega_F}(t)'\cdot H(t)\right]\cdot d\underline{W}^{\mathcal{P}}(t). \end{aligned} \tag{20}$$

3.2 The HJB equation

The individual's portfolio selection problem consists in choosing a policy for investment and consumption, i.e. choosing the controls $\left(\underline{\omega_S}(t),\underline{\omega_F}(t),C(t)\right)\equiv\left(\underline{\omega_S},\underline{\omega_F},C\right)$, so as to maximize the expected utility from consumption, subject to the budget constraint (20). In other words, the representative agent has to find $\left(\underline{\omega_S},\underline{\omega_F},C\right)$ such that:[7]

$$J(v,\underline{x},t)=\max_{\left(\underline{\omega_S},\underline{\omega_F},C\right)}K^{\underline{\omega_S},\underline{\omega_F},C}(v,\underline{x},t), \tag{21}$$

where

$$K^{\underline{\omega_S},\underline{\omega_F},C}(v,\underline{x},t)\equiv E_t\left\{\left.\int_t^T u\left[C(s),s\right]ds\right|V(t)=v \text{ and } \underline{X}(t)=\underline{x}\right\},$$

being dv given by Equation (20) and $d\underline{x}$ given by Equation (10).

The Hamilton-Jacobi-Bellman (HJB) equation for the above stochastic optimal control problem is:

$$\begin{aligned} 0 &= \max_{\left(\underline{\omega_S},\underline{\omega_F},C\right)}\phi\left(\underline{\omega_S},\underline{\omega_F},C;v,\underline{x},t\right) \\ &= \max_{\left(\underline{\omega_S},\underline{\omega_F},C\right)}\left\{u(C,t)+\left(L^{\underline{\omega_S},\underline{\omega_F},C}J\right)(v,\underline{x},t)\right\}, \end{aligned} \tag{22}$$

where the Dynkin's operator is equal to[8]

$$\begin{aligned} \left(L^{\underline{\omega_S},\underline{\omega_F},C}J\right)(v,\underline{x},t) &\equiv J_t+\left\{\underline{\omega_S}'\cdot\left[\underline{\mu_S}(t)-r(t)\underline{1}\right]v+\underline{\omega_F}'\cdot\left[\underline{\mu_F}(t)-r(t)\underline{1}\right]v\right. \\ &\quad \left.+r(t)\,v-p(t)\,C(t)\right\}J_v+J_{\underline{x}'}\cdot\left(\bar{a}\cdot\underline{x}+\underline{\bar{b}}\right) \\ &\quad +\frac{v^2J_{vv}}{2}\left[\underline{\omega_S}'\cdot E(t)+\underline{\omega_F}'\cdot H(t)\right]\cdot\left[E(t)'\cdot\underline{\omega_S}+H(t)'\cdot\underline{\omega_F}\right] \end{aligned}$$

$$+\frac{1}{2}tr\left[J_{\underline{x}\,\underline{x}'}\cdot\Sigma\cdot V^{D}(t)\cdot\Sigma'\right]$$
$$+v\left[\underline{\omega_S}'\cdot E(t)+\underline{\omega_F}'\cdot H(t)\right]\cdot\sqrt{V^{D}(t)}\cdot\Sigma'\cdot J_{v\underline{x}},$$

with $J_{\underline{x}}=\frac{\partial J(v,\underline{x},t)}{\partial\underline{x}}$, $J_{\underline{x}\,\underline{x}'}=\frac{\partial^2 J(v,\underline{x},t)}{\partial\underline{x}\partial\underline{x}'}$, and $J_{v\underline{x}}=\frac{\partial^2 J(v,\underline{x},t)}{\partial v\partial\underline{x}}$, subject to the non-negativity restrictions $\omega_{S_i}\geq 0$ $(i=1,\ldots,m)$ and $C\geq 0$, as well as to the boundary condition $J(v,\underline{x},T)=0$.

Using the Kuhn-Tucker Theorem, the necessary and sufficient conditions for the maximization of the function $\phi\left(\underline{\omega_S},\underline{\omega_F},C;v,\underline{x},t\right)$ are:

$$\phi_C=u_C(t)-p(t)J_v\leq 0, \tag{23}$$

$$\left[u_C(t)-p(t)J_v\right]C=0, \tag{24}$$

$$\begin{aligned}\phi_{\underline{\omega_S}} &= \left[E(t)\cdot E(t)'\cdot\underline{\omega_S}+E(t)\cdot H(t)'\cdot\underline{\omega_F}\right]v^2J_{vv}\\ &\quad+\left[\underline{\mu_S}(t)-r(t)\underline{1}\right]vJ_v+vE(t)\cdot\sqrt{V^{D}(t)}\cdot\Sigma'\cdot J_{v\underline{x}}\\ &\leq \underline{0},\end{aligned} \tag{25}$$

$$\underline{\omega_S}'\cdot\phi_{\underline{\omega_S}}=0, \tag{26}$$

and

$$\begin{aligned}\phi_{\underline{\omega_F}} &= \left[H(t)\cdot H(t)'\cdot\underline{\omega_F}+H(t)\cdot E(t)'\cdot\underline{\omega_S}\right]v^2J_{vv}\\ &\quad+\left[\underline{\mu_F}(t)-r(t)\underline{1}\right]vJ_v+vH(t)\cdot\sqrt{V^{D}(t)}\cdot\Sigma'\cdot J_{v\underline{x}}\\ &= \underline{0}.\end{aligned} \tag{27}$$

4 The equilibrium instantaneous nominal risk-free interest rate

As in Cox, Ingersoll, and Ross (1985a), equilibrium is defined by a set of stochastic processes $\left(r(t),\underline{\mu_F}(t);\underline{\omega_S},\underline{\omega_F},C\right)$ satisfying conditions (23) to (27), as well as the following market clearing conditions:

1. In equilibrium, all wealth is invested in the physical production processes, that is $\underline{\omega_S}'\cdot\underline{1}=1$.
2. In equilibrium, no financial contingent claims are held, i.e. $\underline{\omega_F}=\underline{0}$. That is in equilibrium the net supply or aggregate demand for each financial contingent claim is zero. This is because for each individual who demands some security, there is always another individual that creates and sells it.

The aim of the current Section is to compute, explicitly, an equilibrium formula for $r(t)$, in the context of the Duffie and Kan (1996) model. Initially, a production economy will be used, and the results obtained will be similar to those already generated by Cox, Ingersoll, and Ross (1985a) and Breeden (1986). However, while these authors give equilibrium expressions for the instantaneous *real* risk-free interest rate, here their results will be adapted to the context of a monetary economy. Finally, a one-country pure exchange economy with a cash-in-advance constraint will be used, and a new equilibrium specification for the instantaneous *nominal* riskless interest rate will be obtained.

4.1 The production side of the economy: *a la* Cox, Ingersoll, and Ross (1985a)

Following Cox, Ingersoll, and Ross (1985a), Appendix A expresses $r(t)$ in terms of the indirect utility function:

$$r(t) = \underline{\omega_S}' \cdot \underline{\mu_S}(t) + v\left(\frac{J_{vv}}{J_v}\right)\underline{\omega_S}' \cdot E(t) \cdot E(t)' \cdot \underline{\omega_S} + \underline{\omega_S}' \cdot E(t) \cdot \sqrt{V^D(t)} \cdot \Sigma' \cdot \left(\frac{J_{v\underline{x}}}{J_v}\right). \tag{28}$$

Equation (28) is similar to Equation (14) of Cox, Ingersoll, and Ross (1985a) and to Equation (15) of Breeden (1986). However, it is not exactly equivalent since these last two equations give the equilibrium value of the short-term *real* (not *nominal*) interest rate, which is stated in terms of the *real* wealth, because both models use the single physical good as the numeraire.

Next, $r(t)$ will be derived as an explicit function of the utility of consumption, and no longer as a function of the utility of wealth.

4.2 The consumption side of the economy: *a la* Breeden (1986)

To prove that the equilibrium instantaneous interest rate is equal to minus the expected rate of change in the marginal utility of *nominal* wealth,

$$r(t) = -\frac{\mu_{J_v}(t)}{J_v(v, \underline{x}, t)}, \tag{29}$$

the approach followed by Cox, Ingersoll, and Ross (1985a, Theorem 1) is used in Appendix B.

On the other hand, if condition (24) is considered, while assuming that $C \neq 0$, then

$$J_v(v, \underline{x}, t) = \frac{u_C(t)}{p(t)}. \tag{30}$$

Using Itô's lemma,

$$\mu_{J_v}(t) = \frac{1}{p(t)}\mu_{u_C}(t) - \frac{u_C(t)}{p(t)^2}\mu_p(t) + \frac{1}{2}\frac{2u_C(t)\,p(t)}{p(t)^4}\underline{\sigma_p}(t)' \cdot \underline{\sigma_p}(t) - \frac{1}{p(t)^2}COV\left[du_C(t), dp(t)\right], \tag{31}$$

where

$$du_C(t) = \mu_{u_C}(t)\,dt + \underline{\sigma_{u_C}}(t)' \cdot d\underline{W}^{\mathcal{P}}(t),$$

$u_C(t)$ is the time-t marginal utility of consumption, with $\mu_{u_C}(t) \in \Re$ and $\underline{\sigma_{u_C}}(t) \in \Re^n$,

$$dp(t) = \mu_p(t)\,dt + \underline{\sigma_p}(t)' \cdot d\underline{W}^{\mathcal{P}}(t),$$

$\frac{\mu_p(t)}{p(t)} \in \Re$ represents the time-t expected rate of inflation, and $\underline{\sigma_p}(t) \in \Re^n$. Combining Equations (29), (30), and (31):

$$r(t) = -\frac{\mu_{u_C}(t)}{u_C(t)} + \left\{\frac{\mu_p(t)}{p(t)} - \frac{\underline{\sigma_p}(t)' \cdot \underline{\sigma_p}(t)}{p(t)^2} + \frac{COV\left[du_C(t), dp(t)\right]}{u_C(t)\,p(t)}\right\}. \tag{32}$$

From Breeden (1986, Equation 19), it is known that the first term on the right-hand-side of the previous equation represents the time-t *real* risk-free instantaneous interest rate, which will be denominated by $k(t)$. In order to compute $\left[-\frac{\mu_{u_C}(t)}{u_C(t)}\right]$ explicitly, the following stochastic differential equation for aggregate consumption will be considered:

$$\frac{dC(t)}{C(t)} = \mu_C(t)\,dt + \underline{\sigma_C}(t)' \cdot d\underline{W}^{\mathcal{P}}(t),$$

where $\mu_C(t) \in \Re$ and $\underline{\sigma_C}(t) \in \Re^n$. Applying Itô's lemma to the marginal utility of consumption, it is possible to derive the functional form of its drift:

$$\mu_{u_C} = u_{CC}(t)\,\mu_C(t)\,C(t) + u_{Ct}(t) + \frac{1}{2}u_{CCC}(t)\,\underline{\sigma_C}(t)' \cdot \underline{\sigma_C}(t)\,C(t)^2. \tag{33}$$

Substituting (33) into the first term in the right-hand-side of (32), one obtains the consumption-based equilibrium Equation (22) of Breeden (1986) for the *real* risk-free instantaneous interest rate:

$$\begin{aligned} k(t) &= -\frac{\mu_{u_C}(t)}{u_C(t)} \\ &= -\frac{u_{Ct}(t)}{u_C(t)} - \frac{C(t)\,u_{CC}(t)}{u_C(t)}\mu_C(t) - \frac{1}{2}\frac{C^2(t)\,u_{CCC}(t)}{u_C(t)}\left[\underline{\sigma_C}(t)' \cdot \underline{\sigma_C}(t)\right]. \end{aligned} \tag{34}$$

From now on, it will be considered, as an additional assumption, that the preferences are time-separable, i.e.

A.12) $u(C,t) = e^{-\rho t} U(C)$, where ρ is the constant discount factor or time-preference parameter, $U_C > 0$ (nonsatiation assumption), and $U_{CC} < 0$ (risk aversion assumption).

Therefore, $-\frac{u_{Ct}(t)}{u_C(t)} = -\frac{-\rho e^{-\rho t} U_C}{e^{-\rho t} U_C} = \rho$, and combining Equations (32) and (34),

$$r(t) = k(t) + \left\{ \frac{\mu_p(t)}{p(t)} - \frac{\underline{\sigma_p}(t)' \cdot \underline{\sigma_p}(t)}{p(t)^2} + \frac{COV[du_C(t), dp(t)]}{u_C(t) p(t)} \right\}, \tag{35}$$

with

$$k(t) = \rho - \frac{C(t) u_{CC}(t)}{u_C(t)} \mu_C(t) - \frac{1}{2} \frac{C^2(t) u_{CCC}(t)}{u_C(t)} \left[\underline{\sigma_C}(t)' \cdot \underline{\sigma_C}(t) \right]. \tag{36}$$

The above expression for $r(t)$ is distinct both from Equation (45) of Heston (1988) and from Equation (60) of Cox, Ingersoll, and Ross (1985b). In opposition with Heston (1988), Equation (35) does not correspond to the well known *Fisher identity*, because

$$\begin{aligned} \frac{COV[du_C(t), dp(t)]}{u_C(t) p(t)} &= \frac{\underline{\sigma_{u_C}}(t)' \cdot \underline{\sigma_p}(t)}{u_C(t) p(t)} \\ &= \frac{C(t) u_{CC}(t)}{u_C(t) p(t)} COV \left[\frac{dC(t)}{C(t)}, dp(t) \right] \\ &\neq 0, \end{aligned}$$

that is since we are not assuming *money neutrality* (i.e. it is not assumed that the price level has no effect on the real side of the economy). In fact, Sun (1992) found a significant correlation between the price level and the growth rate of consumption, which does not support the *money neutrality* assumption. On the other hand, Equation (35) shows two important differences when compared to Equation (60) of Cox, Ingersoll, and Ross (1985b). Firstly, because Equation (35) is expressed in terms of the direct utility function, and not in terms of the utility of wealth. Secondly, because in Equation (35) both the price level, $p(t)$, and the expected rate of inflation, $\frac{\mu_p(t)}{p(t)}$, are endogenously determined, and thus one can be sure that they will be consistent with our general equilibrium framework.

4.3 A one-country pure exchange economy

Assuming A.5, and since in equilibrium $\underline{\omega_S}' \cdot \underline{1} = 1$ as well as $\underline{\omega_F} = \underline{0}$, one moves from a production economy to a Lucas (1978) type of pure exchange economy where all output is consumed, that is $C(t) = q(t)$.[9] Hence, Equations (35) and (36) can be stated in terms of the exogenous aggregate output, which means that it is not necessary to solve the HJB equation (22) for the endogenous consumption process:

$$r(t) = k(t) + \left[\frac{\mu_p(t)}{p(t)} - \frac{\underline{\sigma_p}(t)' \cdot \underline{\sigma_p}(t)}{p(t)^2} + \frac{q(t)\, u_{qq}(t)}{u_q(t)} \underline{\sigma_q}(t)' \cdot \frac{\underline{\sigma_p}(t)}{p(t)} \right], \tag{37}$$

with

$$k(t) = \rho - \frac{q(t)\, u_{qq}(t)}{u_q(t)} \mu_q(t) - \frac{1}{2} \frac{q(t)^2\, u_{qqq}(t)}{u_q(t)} \left[\underline{\sigma_q}(t)' \cdot \underline{\sigma_q}(t) \right]. \tag{38}$$

Equation (38) corresponds to Equation (11) of Bakshi and Chen (1997a). The next Theorem rewrites the above equilibrium solution for the *nominal* short-term interest rate only in terms of the exogenous output and money supply processes.

Theorem 1 *In equilibrium, the instantaneous nominal interest rate is*

$$\begin{aligned} r(t) &= \left[\rho + \mu_M(t) - \mu_q(t) - \underline{\sigma_M}(t)' \cdot \underline{\sigma_M}(t) + \sigma_{q,M}(t) \right] \\ &\quad - \frac{q(t)\, u_{qq}(t)}{u_q(t)} \left[\mu_q(t) - \sigma_{q,M}(t) + \underline{\sigma_q}(t)' \cdot \underline{\sigma_q}(t) \right] \\ &\quad - \frac{1}{2} \frac{q(t)^2\, u_{qqq}(t)}{u_q(t)} \left[\underline{\sigma_q}(t)' \cdot \underline{\sigma_q}(t) \right], \end{aligned} \tag{39}$$

where $\sigma_{q,M}(t) \equiv COV\left[\frac{dq(t)}{q(t)}, \frac{dM(t)}{M(t)} \right]$.

Proof. Applying Itô's lemma to $p(t) = \dfrac{M(t)}{q(t)}$, all terms in Equation (37) can be expressed as functions of only $q(t)$ and $M(t)$:

$$\mu_p(t) = \frac{\mu_M(t)\, M(t)}{q(t)} - \frac{M(t)\, \mu_q(t)}{q(t)} + \frac{M(t)\, \underline{\sigma_q}(t)' \cdot \underline{\sigma_q}(t)}{q(t)} - \frac{\underline{\sigma_q}(t)' \cdot \underline{\sigma_M}(t)\, M(t)}{q(t)},$$

$$\frac{\mu_p(t)}{p(t)} = \mu_M(t) - \mu_q(t) + \underline{\sigma_q}(t)' \cdot \underline{\sigma_q}(t) - \underline{\sigma_q}(t)' \cdot \underline{\sigma_M}(t),$$

$$\underline{\sigma_p}(t) = \frac{1}{q(t)} \underline{\sigma_M}(t)\, M(t) - \frac{M(t)}{q(t)^2} \underline{\sigma_q}(t)\, q(t),$$

$$\frac{\underline{\sigma_p}(t)' \cdot \underline{\sigma_p}(t)}{p(t)^2} = \left[\underline{\sigma_M}(t) - \underline{\sigma_q}(t) \right]' \cdot \left[\underline{\sigma_M}(t) - \underline{\sigma_q}(t) \right],$$

and

$$\frac{\underline{\sigma_q}(t)' \cdot \underline{\sigma_p}(t)}{p(t)} = \underline{\sigma_q}(t)' \cdot \left[\underline{\sigma_M}(t) - \underline{\sigma_q}(t) \right].$$

Hence,

$$r(t) = \rho + \left[\mu_M(t) - \mu_q(t) + \underline{\sigma_q}(t)' \cdot \underline{\sigma_q}(t) - \underline{\sigma_q}(t)' \cdot \underline{\sigma_M}(t) \right]$$

$$-\left[\underline{\sigma_M}(t)' \cdot \underline{\sigma_M}(t) - 2\underline{\sigma_q}(t)' \cdot \underline{\sigma_M}(t) + \underline{\sigma_q}(t)' \cdot \underline{\sigma_q}(t)\right]$$
$$-\frac{q(t)\, u_{qq}(t)}{u_q(t)}\left[\mu_q(t) - \underline{\sigma_q}(t)' \cdot \underline{\sigma_M}(t) + \underline{\sigma_q}(t)' \cdot \underline{\sigma_q}(t)\right]$$
$$-\frac{1}{2}\frac{q(t)^2\, u_{qqq}(t)}{u_q(t)}\left[\underline{\sigma_q}(t)' \cdot \underline{\sigma_q}(t)\right],$$

which yields Equation (39) after collecting alike terms. Following Bakshi and Chen (1997b, pages 818-819), an alternative derivation of Equation (39) is presented in Appendix C. ■

According to (39), the short-term *nominal* equilibrium interest rate is increasing in: the time-preference parameter; the expected growth rate of money supply; the expected rate of change in the aggregate output (if the coefficient of relative risk aversion is greater than one); and in the volatility of the aggregate output growth rate. On the other hand, $r(t)$ is decreasing in: the volatility of the money supply growth rate; and in the covariance between the growth rates of aggregate output and money supply (again, if $-\frac{q(t)u_{qq}(t)}{u_q(t)} > 1$).

Equations (28) and (39) generate the same term structure of interest rates, because they must hold simultaneously in equilibrium. However, the use of Equation (28) requires the existence of a closed-form solution for the indirect utility function, which has to be obtained by solving the HJB equation (22), or requires the assumption of restrictive preferences: namely, the use of a log utility function, as is the case in Cox, Ingersoll, and Ross (1985a) and Longstaff and Schwartz (1992a). Consequently, next sections will try to fit the Duffie and Kan (1996) model into a general equilibrium framework with more realistic assumptions about preferences than those implied by a Bernoulli logarithmic utility function, through the use of Equation (39) instead of Equation (28). In fact, it turns out to be easy to work with Equation (39) since the stochastic processes for the aggregate output and for the money supply can be exogenously specified in a suitable fashion.

5 The equilibrium factor risk premiums

In order to fit the Duffie and Kan (1996) model into a general equilibrium framework, it is necessary to prove that our general equilibrium assumptions imply an affine form for $r(t)$ -as in Equation (4)- and a risk-adjusted process for $\underline{X}(t)$ equivalent to the stochastic differential equation (5). But, the derivation of the equilibrium risk-neutral process for the model' factors (that is consistent with our general equilibrium setup), requires the computation of the risk premiums associated with each one of the non-traded state variables. Only after having derived such factor risk premiums, it is then possible to specify the equilibrium risk-adjusted drift for $d\underline{X}(t)$, by applying Girsanov's Theorem or by obtaining the PDE that must be satisfied, in equilibrium, by any interest rate contingent claim.

In equilibrium, since $\underline{\omega_F} = \underline{0}$, Equation (27) becomes:

$$vJ_v \left[\underline{\mu_F}(t) - r(t)\underline{1}\right] + v^2 J_{vv} H(t) \cdot E(t)' \cdot \underline{\omega_S} + vH(t) \cdot \sqrt{V^D(t)} \cdot \Sigma' \cdot J_{v\underline{x}} = \underline{0},$$

that is

$$\left[\underline{\mu_F}(t) - r(t)\underline{1}\right] = -v\left(\frac{J_{vv}}{J_v}\right) H(t) \cdot E(t)' \cdot \underline{\omega_S} - H(t) \cdot \sqrt{V^D(t)} \cdot \Sigma' \cdot \left(\frac{J_{v\underline{x}}}{J_v}\right).$$

Both sides of the above equation are $n \times 1$ matrices. Taking just their i^{th}-row,

$$\mu_{F_i}(t) - r(t) = -v\left(\frac{J_{vv}}{J_v}\right) \underline{h_i}(t)' \cdot E(t)' \cdot \underline{\omega_S} - \underline{h_i}(t)' \cdot \sqrt{V^D(t)} \cdot \Sigma' \cdot \left(\frac{J_{v\underline{x}}}{J_v}\right), \tag{40}$$

where $\mu_{F_i}(t)$ is the expected *nominal time-t* rate of return on the i^{th} financial contingent claim, $\left[\mu_{F_i}(t) - r(t)\right]$ represents the equilibrium expected excess *nominal* rate of return (over the risk-free interest rate) generated by the i^{th} financial contingent claim, and $\underline{h_i}(t)'$ is the i^{th}-row of matrix $H(t)$.

In order to obtain $\underline{h_i}(t)$ explicitly, Itô's lemma will be applied to the value of the i^{th} financial contingent claim, $F_i(\underline{x}, t)$, where it is assumed that the contractual terms of the financial contingent claim do not depend explicitly on wealth (and, again, only time-dependencies will be retained):

$$dF_i(t) = F_i(t)\,\mu_{F_i}(t)\,dt + \frac{\partial F_i(t)}{\partial \underline{x}'} \cdot \Sigma \cdot \sqrt{V^D(t)} \cdot d\underline{W^{\mathcal{P}}}(t). \tag{41}$$

Comparing Equations (14) and (41), it follows that

$$\begin{aligned} F_i(t)\,\underline{h_i}(t)' &= \frac{\partial F_i(t)}{\partial \underline{x}'} \cdot \Sigma \cdot \sqrt{V^D(t)} \\ &\equiv \underline{\sigma_{F_i}}(t)'. \end{aligned}$$

Thus, Equation (40) is equivalent to:

$$\begin{aligned} \left[\mu_{F_i}(t) - r(t)\right] F_i(t) &= -v\left(\frac{J_{vv}}{J_v}\right) \frac{\partial F_i(t)}{\partial \underline{x}'} \cdot \Sigma \cdot \sqrt{V^D(t)} \cdot E(t)' \cdot \underline{\omega_S} \\ &\quad - \frac{\partial F_i(t)}{\partial \underline{x}'} \cdot \Sigma \cdot V^D(t) \cdot \Sigma' \cdot \left(\frac{J_{v\underline{x}}}{J_v}\right). \end{aligned} \tag{42}$$

On the other hand, Equation (41) and the stochastic process - Equation (89)- followed by the marginal utility of wealth, imply that

$$
\begin{aligned}
& COV\left[dF_i\left(\underline{x},t\right),dJ_v\left(v,\underline{x},t\right)\right] \\
= \quad & vJ_{vv}\frac{\partial F_i\left(t\right)}{\partial \underline{x}'}\cdot\Sigma\cdot\sqrt{V^D\left(t\right)}\cdot E\left(t\right)'\cdot\underline{\omega_S}+\frac{\partial F_i\left(t\right)}{\partial \underline{x}'}\cdot\Sigma\cdot V^D\left(t\right)\cdot\Sigma'\cdot J_{v\underline{x}}.
\end{aligned}
\tag{43}
$$

Comparing Equations (42) and (43), a similar result to Cox, Ingersoll, and Ross (1985a, Equation 27) is obtained, the only difference being the fact that one is now considering expected excess equilibrium *nominal* returns instead of *real* ones:

$$
\left[\mu_{F_i}\left(t\right)-r\left(t\right)\right]F_i\left(t\right)=-\frac{1}{J_v\left(v,\underline{x},t\right)}COV\left[dF_i\left(\underline{x},t\right),dJ_v\left(v,\underline{x},t\right)\right]. \tag{44}
$$

However, since a solution for the indirect utility function is not available, the above expression is of little practical use. In order to compute the equilibrium risk premiums required for the i^{th} financial contingent claim, as a function of estimable parameters, it is necessary to convert the right-hand-side of Equation (44) in terms of the exogenously specified output and money supply processes. Such task is accomplished by the following Theorem.

Theorem 2 *In equilibrium, the factor risk premiums on any financial contingent claim $F\left(t\right)$ satisfy*

$$
\begin{aligned}
\left[\mu_F\left(t\right)-r\left(t\right)\right]F\left(t\right)= \quad & -\left[1+\frac{q\left(t\right)u_{qq}\left(t\right)}{u_q\left(t\right)}\right]COV\left[dF\left(t\right),\frac{dq\left(t\right)}{q\left(t\right)}\right] \\
& +COV\left[dF\left(t\right),\frac{dM\left(t\right)}{M\left(t\right)}\right].
\end{aligned}
\tag{45}
$$

Proof. As a first step, condition $J_v\left(v,\underline{x},t\right)=\frac{u_C(t)}{p(t)}$ implies that Equation (44) can be rewritten as

$$
\left[\mu_{F_i}\left(t\right)-r\left(t\right)\right]F_i\left(t\right)=-\frac{p\left(t\right)}{u_C\left(t\right)}COV\left[dF_i\left(t\right),d\left(\frac{u_C}{p}\right)\left(t\right)\right].
$$

From Itô's lemma, the diffusion of the stochastic process $d\left(\frac{u_C}{p}\right)\left(t\right)$ is given by $\frac{1}{p(t)}\underline{\sigma_{u_C}}\left(t\right)'-\frac{u_C(t)}{p(t)^2}\underline{\sigma_p}\left(t\right)'$, and therefore

$$
\left[\mu_{F_i}\left(t\right)-r\left(t\right)\right]F_i\left(t\right)=-\frac{1}{u_C\left(t\right)}\underline{\sigma_{F_i}}\left(t\right)'\cdot\underline{\sigma_{u_C}}\left(t\right)+\frac{1}{p\left(t\right)}\underline{\sigma_{F_i}}\left(t\right)'\cdot\underline{\sigma_p}\left(t\right).
$$

Applying again Itô's lemma while considering Equation (10) and the equilibrium budget constraint -Equation (88)- it follows that

$$\underline{\sigma_{u_C}}(t)' = u_{CC}(t)\left[vC_v(t)\underline{\omega_S}' \cdot E(t) + \frac{\partial C(t)}{\partial \underline{x}'} \cdot \Sigma \cdot \sqrt{V^D(t)}\right],$$

and

$$\underline{\sigma_C}(t)' = \frac{vC_v(t)\underline{\omega_S}' \cdot E(t) + \frac{\partial C(t)}{\partial \underline{x}'} \cdot \Sigma \cdot \sqrt{V^D(t)}}{C(t)}.$$

Hence $\underline{\sigma_{u_C}}(t) = C(t)\, u_{CC}(t)\, \underline{\sigma_C}(t)$, and because $C(t) = q(t)$, then

$$\left[\mu_{F_i}(t) - r(t)\right] F_i(t) = -\frac{q(t)\, u_{qq}(t)}{u_q(t)} \underline{\sigma_{F_i}}(t)' \cdot \underline{\sigma_q}(t) + \frac{1}{p(t)} \underline{\sigma_{F_i}}(t)' \cdot \underline{\sigma_p}(t).$$

Moreover, since $\underline{\sigma_p}(t) = p(t)\,\underline{\sigma_M}(t) - p(t)\,\underline{\sigma_q}(t)$,

$$\left[\mu_{F_i}(t) - r(t)\right] F_i(t) = -\frac{q(t)\, u_{qq}(t)}{u_q(t)} \underline{\sigma_{F_i}}(t)' \cdot \underline{\sigma_q}(t) + \underline{\sigma_{F_i}}(t)' \cdot \underline{\sigma_M}(t) - \underline{\sigma_{F_i}}(t)' \cdot \underline{\sigma_q}(t).$$

Finally, applying the above equation to a general financial contingent claim with a value of $F(t)$ and an expected rate of return of $\mu_F(t)$, the equilibrium solution (45) follows. An alternative derivation is provided in Appendix D. ■

Thus, to find the equilibrium factor risk premiums (as well as the instantaneous *nominal* spot equilibrium interest rate) for the Duffie and Kan (1996) model, it is just necessary to specify an utility function as well as suitable output and money supply stochastic processes.

Before proceeding, three remarks should be made. First, Equation (45) implies that the factor risk premiums are increasing in the conditional covariance of the contingent claim value with: *i*) the rate of change in the aggregate output (if the coefficient of relative risk aversion is greater than one); and, with *ii*) the growth rate of money supply. In other words, Equation (45) shows that both "production risk" (i.e. technological shocks) and "monetary risk" (that is, inflationary shocks) matter. Second, from Cox, Ingersoll, and Ross (1985a, equation 30) or from Equation (9) of Bakshi and Chen (1997a), it is well known that the equilibrium expected excess *real* rate of return is equal to $\left(-\frac{q(t)u_{qq}(t)}{u_q(t)}\right) COV\left[dF(t), \frac{dq(t)}{q(t)}\right]$. Subtracting this "*real* risk" compensation from Equation (45), it can now be concluded that the equilibrium compensation for "*nominal* risk" must be given by $COV\left[dF(t), \frac{dM(t)}{M(t)}\right] - COV\left[dF(t), \frac{dq(t)}{q(t)}\right]$. Thirdly, Equation (45) also shows that even in a risk-neutral economy -where $-\frac{q(t)u_{qq}(t)}{u_q(t)} = 0$, i.e. with a linear utility function-the equilibrium expected excess *nominal* rate of return on a financial contingent claim would still be non-zero (unless $COV\left[dF(t), \frac{dM(t)}{M(t)}\right] = COV\left[dF(t), \frac{dq(t)}{q(t)}\right]$). This means that to derive a Duffie and Kan (1996) model specification under the original probability measure $\mathcal{P}$ that is compatible with the specification given by the authors under the equivalent martingale measure $\mathcal{Q}$, it would be unrealistic to assume a zero vector of market prices of risk, since such assumption would most probably be inconsistent with our general equilibrium setup.

6 The Duffie and Kan (1996) model in a constant relative risk aversion economy

In order to obtain the Duffie and Kan (1996) model from our general equilibrium framework, assumptions A.5, A.6, and A.12 must be further specialized.

6.1 An economy with a power utility function

Now, an economy with decreasing absolute risk aversion will be considered, and more specifically, a power utility function will be used to characterize the preferences of the representative investor. Hence, assumption A.12 is specialized into:

A.12')

$$u(C,t) = e^{-\rho t}\frac{C^{\gamma}-1}{\gamma}, \tag{46}$$

where $\gamma < 1$ (and thus $u_{CC}(t) < 0$), $\gamma \neq 0$, and $(1-\gamma)$ is the Pratt's measure of relative risk aversion.[10]

Since $C(t) = q(t)$, and using (46), then $u(q,t) = e^{-\rho t}\frac{q^{\gamma}-1}{\gamma}$,

$$-\frac{q(t)\, u_{qq}(t)}{u_q(t)} = 1-\gamma, \tag{47}$$

and

$$\frac{q(t)^2\, u_{qqq}(t)}{u_q(t)} = (\gamma-1)(\gamma-2). \tag{48}$$

The choice of the utility function under use was not intended to be the most general one possible but rather as general as necessary to nest, as special cases, all the affine general equilibrium interest rate frameworks presented so far in the literature (which are invariably based on the more restrictive log utility function). Nevertheless, it can be easily shown that the power utility function considered hereafter is the most general specification, under the hyperbolic absolute risk aversion class,[11] that generates constant (i.e. output independent) values for both quantities $-\frac{q(t)u_{qq}(t)}{u_q(t)}$ and $\frac{q(t)^2 u_{qqq}(t)}{u_q(t)}$ appearing in expressions (39) and (45), and therefore that supports the Duffie and Kan (1996) model under an affine specification for both the drifts and the instantaneous variances of the aggregate output and money supply processes.

In order to derive a Duffie and Kan (1996) model from our general equilibrium setup, the stochastic processes for the aggregate output and for the money supply (i.e. the functional form of $\mu_q(t)$, $\underline{\sigma_q}(t)$, $\mu_M(t)$, and $\underline{\sigma_M}(t)$) must be defined in such a way that two conditions are met: *i)* $r(t)$ must be an affine function of the state variables; and *ii)* $\underline{\mu}[\underline{X}(t)]$ must also be affine.

From Theorem 1, condition *i)* implies that $\mu_M(t)$, $\mu_q(t)$, $[\underline{\sigma_M}(t)' \cdot \underline{\sigma_M}(t)]$, $\left[\underline{\sigma_q}(t)' \cdot \underline{\sigma_q}(t)\right]$, and $\sigma_{q,M}(t)$ must all be affine functions of $\underline{X}(t)$. So, the drifts of the stochastic processes (12) and (13) can be defined as:

$$\mu_q(t) = \eta + \underline{\theta}' \cdot \underline{X}(t), \tag{49}$$

and

$$\mu_M(t) = \pi + \underline{\phi}' \cdot \underline{X}(t), \tag{50}$$

where $\eta, \pi \in \Re$, and $\underline{\theta}, \underline{\phi} \in \Re^n$.

Considering condition *ii)*, since $\underline{\mu}[\underline{X}(t)] = \underline{v}[\underline{X}(t)] - \sigma[\underline{X}(t)] \cdot \underline{\Lambda}[\underline{X}(t)]$ and because $\underline{v}[\underline{X}(t)]$ is defined by Equation (10) as an affine function of the state vector, then $\underline{\mu}[\underline{X}(t)]$ can only be affine if $\sigma[\underline{X}(t)] \cdot \underline{\Lambda}[\underline{X}(t)]$ is also affine. But, because $[\mu_F(t) - r(t)]F(t) = \underline{\sigma_F}(t)' \cdot \underline{\Lambda}[\underline{X}(t)]$, with $\underline{\sigma_F}(t)' = \frac{\partial F(t)}{\partial \underline{x}'} \cdot \sigma[\underline{X}(t)]$, and[12]

$$[\mu_F(t) - r(t)]F(t) = \frac{\partial F(t)}{\partial \underline{x}'} \cdot \left[-\gamma\Sigma \cdot \sqrt{V^D(t)} \cdot \underline{\sigma_q}(t) + \Sigma \cdot \sqrt{V^D(t)} \cdot \underline{\sigma_M}(t)\right], \tag{51}$$

then

$$\sigma[\underline{X}(t)] \cdot \underline{\Lambda}[\underline{X}(t)] = -\gamma\Sigma \cdot \sqrt{V^D(t)} \cdot \underline{\sigma_q}(t) + \Sigma \cdot \sqrt{V^D(t)} \cdot \underline{\sigma_M}(t),$$

and thus $\underline{\mu}[\underline{X}(t)]$ is affine if and only if $\left[\Sigma \cdot \sqrt{V^D(t)} \cdot \underline{\sigma_q}(t)\right]$ and $\left[\Sigma \cdot \sqrt{V^D(t)} \cdot \underline{\sigma_M}(t)\right]$ are both affine functions of $\underline{X}(t)$. But, this is only possible if $\underline{\sigma_q}(t)$ and $\underline{\sigma_M}(t)$ are both equal to:

1. $\sqrt{V^D(t)}$ multiplied by some $n \times 1$ vector of parameters, since $V^D(t)$ is affine; or

2. $\left(\sqrt{V^D(t)}\right)^{-1}$ multiplied by some $n \times 1$ vector of parameters, since a constant is also an affine function; or even

3. A null $n \times 1$ vector, since zero can also be considered as an affine function.

Although all these three alternatives are possible, the first one will be chosen since it represents the most general case. Thus,

$$\underline{\sigma_q}(t) = \sqrt{V^D(t)} \cdot \underline{\varphi}, \tag{52}$$

and

$$\underline{\sigma_M}(t) = \sqrt{V^D(t)} \cdot \underline{\chi}, \tag{53}$$

where $\underline{\varphi} \in \Re^n$ has φ_i as its i^{th} element, and $\underline{\chi} \in \Re^n$ contains χ_i as its i^{th} element. Equations (52) and (53) allow us to respect not only condition $ii)$ but also condition $i)$, since $\left[\underline{\sigma_M}(t)' \cdot \underline{\sigma_M}(t)\right] = \underline{\chi}' \cdot V^D(t) \cdot \underline{\chi}$, $\left[\underline{\sigma_q}(t)' \cdot \underline{\sigma_q}(t)\right] = \underline{\varphi}' \cdot V^D(t) \cdot \underline{\varphi}$, and $\sigma_{q,M}(t) = \underline{\chi}' \cdot V^D(t) \cdot \underline{\varphi}$ are all affine functions of $\underline{X}(t)$.

Combining Equations (49) with (52), and (50) with (53), assumptions A.5 and A.6 are specialized into:

A.5')

$$\frac{dq(t)}{q(t)} = \left[\eta + \underline{\theta}' \cdot \underline{X}(t)\right] dt + \underline{\varphi}' \cdot \sqrt{V^D(t)} \cdot d\underline{W^{\mathcal{P}}}(t). \tag{54}$$

A.6')

$$\frac{dM(t)}{M(t)} = \left[\pi + \underline{\phi}' \cdot \underline{X}(t)\right] dt + \underline{\chi}' \cdot \sqrt{V^D(t)} \cdot d\underline{W^{\mathcal{P}}}(t). \tag{55}$$

To prove that our general equilibrium framework generates a Duffie and Kan (1996) model, it is only necessary to show that assumptions A.5', A.6', and A.12' allow us to: $i)$ Specialize Equation (39) into Equation (4); and $ii)$ Define a risk-adjusted process for $\underline{X}(t)$ equivalent to Equation (5). Next Theorem verifies requirement $i)$.

Theorem 3 *In a Duffie and Kan (1996) general equilibrium model with a power utility function, and with output and money supply processes described by assumptions A.5' and A.6', respectively, the equilibrium specification for the instantaneous nominal spot interest rate is given by:*

$$r(t) = f + \underline{G}' \cdot \underline{X}(t), \tag{56}$$

with

$$f = \rho + \pi - \gamma\eta + \left[\gamma\underline{\Gamma} - \underline{\chi}^2 + \frac{\gamma(1-\gamma)}{2}\underline{\varphi}^2\right]' \cdot \underline{\alpha},$$

and

$$\underline{G} = \underline{\phi} - \gamma\underline{\theta} + \beta \cdot \left[\gamma\underline{\Gamma} - \underline{\chi^2} + \frac{\gamma(1-\gamma)}{2}\underline{\varphi^2}\right],$$

where $\underline{\varphi^2}, \underline{\chi^2}, \underline{\Gamma} \in \Re^n$ *possess* $(\varphi_i)^2$, $(\chi_i)^2$, *and* $(\chi_i\varphi_i)$ *as their* i^{th} *element, respectively.*

Proof. Substituting (47), (48), (49), (50), (52), and (53) into Equation (39),

$$\begin{aligned} r(t) &= \rho + \pi + \underline{\phi}' \cdot \underline{X}(t) - \left[\eta + \underline{\theta}' \cdot \underline{X}(t)\right] - \underline{\chi}' \cdot V^D(t) \cdot \underline{\chi} + \underline{\chi}' \cdot V^D(t) \cdot \underline{\varphi} \\ &\quad + (1-\gamma)\left[\eta + \underline{\theta}' \cdot \underline{X}(t) - \underline{\chi}' \cdot V^D(t) \cdot \underline{\varphi} + \underline{\varphi}' \cdot V^D(t) \cdot \underline{\varphi}\right] \\ &\quad - \frac{(\gamma-1)(\gamma-2)}{2}\underline{\varphi}' \cdot V^D(t) \cdot \underline{\varphi}. \end{aligned} \tag{57}$$

But, because $\underline{\varphi}' \cdot V^D(t) \cdot \underline{\varphi} = \sum_{i=1}^{n} \varphi_i^2 v_i(t)$, and since $v_i(t) = \alpha_i + \underline{\beta_i}' \cdot \underline{X}(t)$, then $\underline{\varphi}' \cdot V^D(t) \cdot \underline{\varphi} = \sum_{i=1}^{n} \varphi_i^2 \alpha_i + \sum_{i=1}^{n} \varphi_i^2 \underline{\beta_i}' \cdot \underline{X}(t)$, i.e.

$$\underline{\varphi}' \cdot V^D(t) \cdot \underline{\varphi} = \left(\underline{\varphi^2}\right)' \cdot \underline{\alpha} + \left(\underline{\varphi^2}\right)' \cdot \beta' \cdot \underline{X}(t), \tag{58}$$

where $\underline{\varphi^2} \in \Re^n$ has $(\varphi_i)^2$ as its i^{th} component, α_i is the i^{th} element of $\underline{\alpha}$, and β is a $n \times n$ matrix whose i^{th}-column is $\underline{\beta_i}$. Similarly, it is easy to show that

$$\underline{\chi}' \cdot V^D(t) \cdot \underline{\chi} = \left(\underline{\chi^2}\right)' \cdot \underline{\alpha} + \left(\underline{\chi^2}\right)' \cdot \beta' \cdot \underline{X}(t), \tag{59}$$

and

$$\underline{\chi}' \cdot V^D(t) \cdot \underline{\varphi} = \underline{\Gamma}' \cdot \underline{\alpha} + \underline{\Gamma}' \cdot \beta' \cdot \underline{X}(t), \tag{60}$$

where $\underline{\chi^2} \in \Re^n$ has $(\chi_i)^2$ as its i^{th} component, and $(\chi_i\varphi_i)$ is the i^{th} element of $\underline{\Gamma} \in \Re^n$. Equations (58), (59), and (60) prove that assumptions A.5' and A.6' guarantee affine specifications for $\left[\underline{\sigma_q}(t)' \cdot \underline{\sigma_q}(t)\right]$, $\left[\underline{\sigma_M}(t)' \cdot \underline{\sigma_M}(t)\right]$, and $\sigma_{q,M}(t)$.

Combining the last four equations,

$$\begin{aligned} r(t) &= \Big\{\rho + \pi - \eta + (1-\gamma)\eta + \big[-\underline{\chi^2} + \underline{\Gamma} - (1-\gamma)\underline{\Gamma} + (1-\gamma)\underline{\varphi^2} \\ &\quad - \frac{(\gamma-1)(\gamma-2)}{2}\underline{\varphi^2}\big]' \cdot \underline{\alpha}\Big\} \\ &\quad + \Big[\underline{\phi}' - \underline{\theta}' - \left(\underline{\chi^2}\right)' \cdot \beta' + \underline{\Gamma}' \cdot \beta' + (1-\gamma)\underline{\theta}' - (1-\gamma)\underline{\Gamma}' \cdot \beta' \\ &\quad + (1-\gamma)\left(\underline{\varphi^2}\right)' \cdot \beta' - \frac{(\gamma-1)(\gamma-2)}{2}\left(\underline{\varphi^2}\right)' \cdot \beta'\Big] \cdot \underline{X}(t), \end{aligned}$$

and simplifying terms, Equation (56) is obtained. ■

Equation (56) shows that our general equilibrium framework provides an affine form for the instantaneous spot risk-free *nominal* interest rate. Moreover, the derivation of Equation (56) also showed that it was only possible to obtain an affine form for $r(t)$ because the drift, the variance, and the covariance of the output and money supply processes were also specified as affine functions of $\underline{X}(t)$.

Theorem 4 proves that it is possible to derive a risk-neutral process for $\underline{X}(t)$ equivalent to Equation (5), and therefore shows that the Duffie and Kan (1996) model is in fact consistent with our type of economy.

Theorem 4 *In a Duffie and Kan (1996) general equilibrium model with a power utility function, and with output and money supply processes described by assumptions A.5' and A.6', respectively:*

1. *The risk-neutral process followed by the state variables under the equivalent martingale measure $\mathcal{Q}$ is equal to*

$$d\underline{X}(t) = [a \cdot \underline{X}(t) + \underline{b}]\, dt + \Sigma \cdot \sqrt{V^D(t)} \cdot d\underline{W}^{\mathcal{Q}}(t),$$

if and only if the stochastic process followed by the state variables under the original probability measure $\mathcal{P}$ is assumed to be given by:

$$d\underline{X}(t) = \left[\bar{a} \cdot \underline{X}(t) + \bar{\underline{b}}\right] dt + \Sigma \cdot \sqrt{V^D(t)} \cdot d\underline{W}^{\mathcal{P}}(t), \tag{61}$$

where

$$\bar{a} = a + \Sigma \cdot \Omega^D \cdot \beta',$$

and

$$\bar{\underline{b}} = \underline{b} + \Sigma \cdot \Omega^D \cdot \underline{\alpha},$$

with

$$\Omega^D = diag\{\chi_1 - \gamma\varphi_1, \ldots, \chi_n - \gamma\varphi_n\}.$$

2. $d\underline{W}^{\mathcal{Q}}(t) = \underline{\Lambda}[\underline{X}(t)]\, dt + d\underline{W}^{\mathcal{P}}(t)$, *with*

$$\underline{\Lambda}[\underline{X}(t)] = \sqrt{V^D(t)} \cdot \left(\underline{\chi} - \gamma\underline{\varphi}\right). \tag{62}$$

Proof. In order to obtain a relation between the risk-neutral and the non-risk adjusted drifts of the model' state variables, it is necessary to compute the Duffie and Kan (1996) model' factor risk premiums (under a CRRA economy). For that purpose, Equations (51), (52), and (53) can be combined into

$$[\mu_F(t) - r(t)]\,F(t) = \frac{\partial F(t)}{\partial \underline{x}'} \cdot \left[-\gamma\Sigma \cdot V^D(t) \cdot \underline{\varphi} + \Sigma \cdot V^D(t) \cdot \underline{\chi}\right], \tag{63}$$

where $\left[\Sigma \cdot V^D(t) \cdot \left(\underline{\chi} - \gamma\underline{\varphi}\right)\right]$ is the vector of factor risk premiums, or the vector Φ_Y in the terminology of Cox, Ingersoll, and Ross (1985a). Because

$$[\mu_F(t) - r(t)]\,F(t) = \frac{\partial F(t)}{\partial \underline{x}'} \cdot \Sigma \cdot \sqrt{V^D(t)} \cdot \underline{\Lambda}\left[\underline{X}(t)\right],$$

Equation (62) follows for the vector of market prices of risk.[13] Equation (63) identifies the analytical formula of the equilibrium risk premium, which makes it possible to derive the fundamental PDE for the Duffie and Kan (1996) model, under a power utility function. Since $F(t)$ is considered to be wealth-independent,

$$\begin{aligned} \mu_F(t)\,F(t) &= (LF)(\underline{x}, t) \\ &= \frac{\partial F(t)}{\partial t} + \frac{\partial F(t)}{\partial \underline{x}'} \cdot \left(\bar{a} \cdot \underline{x} + \underline{\bar{b}}\right) + \frac{1}{2} tr\left[\frac{\partial^2 F(t)}{\partial \underline{x} \partial \underline{x}'} \cdot \Sigma \cdot V^D(t) \cdot \Sigma'\right]. \end{aligned} \tag{64}$$

Combining (63) and (64), the fundamental valuation equation that must be satisfied by the equilibrium value of any financial contingent claim is obtained:

$$\begin{aligned} & \frac{\partial F(t)}{\partial \underline{x}'} \cdot \left(\bar{a} \cdot \underline{x} + \underline{\bar{b}}\right) + \frac{\partial F(t)}{\partial t} + \frac{1}{2} tr\left[\frac{\partial^2 F(t)}{\partial \underline{x} \partial \underline{x}'} \cdot \Sigma \cdot V^D(t) \cdot \Sigma'\right] - r(t)\,F(t) \\ = & \frac{\partial F(t)}{\partial \underline{x}'} \cdot \left[\Sigma \cdot V^D(t) \cdot \left(\underline{\chi} - \gamma\underline{\varphi}\right)\right]. \end{aligned} \tag{65}$$

The right-hand-side of Equation (65) can be simplified, providing a simple expression for the risk-neutral process followed by the model' state variables:

$$\begin{aligned} \Sigma \cdot V^D(t) \cdot \left(\underline{\chi} - \gamma\underline{\varphi}\right) &= \Sigma \cdot \begin{bmatrix} (\chi_1 - \gamma\varphi_1)\, v_1(t) \\ \vdots \\ (\chi_n - \gamma\varphi_n)\, v_n(t) \end{bmatrix} \\ &= \Sigma \cdot \Omega^D \cdot \underline{\alpha} + \Sigma \cdot \Omega^D \cdot \beta' \cdot \underline{X}(t). \end{aligned}$$

Thus, Equation (65) can be rewritten as

$$\begin{aligned} 0 &= \frac{\partial F(t)}{\partial \underline{x}'} \cdot \left[\left(\bar{a} - \Sigma \cdot \Omega^D \cdot \beta'\right) \cdot \underline{x} + \left(\underline{\bar{b}} - \Sigma \cdot \Omega^D \cdot \underline{\alpha}\right)\right] + \frac{\partial F(t)}{\partial t} \\ &\quad + \frac{1}{2} tr\left[\frac{\partial^2 F(t)}{\partial \underline{x} \partial \underline{x}'} \cdot \Sigma \cdot V^D(t) \cdot \Sigma'\right] - r(t)\,F(t), \end{aligned} \tag{66}$$

which, when compared with (7), yields Equation (61). ■

Equations (56) and (61) completely specify our $(P, \underline{\nu}, \underline{\Lambda}, \sigma)$ compatible term structure model (under a power utility function), and prove that the Duffie and Kan (1996) model can in fact be fitted into our general equilibrium framework. Equation (61) can now be used to estimate the model parameters from a time-series of values for the state variables.

6.2 A special case: an economy with a log utility function

Because the log utility function is just a special case of the power utility function (as γ tends to zero), the Duffie and Kan (1996) model can still be fitted into a general equilibrium setup if assumption A.12 is further specialized, maintaining all the other assumptions unchanged:

A.12")

$$u(C, t) = e^{-\rho t} \ln (C) . \tag{67}$$

Next Corollary presents the equilibrium instantaneous *nominal* risk-free interest rate consistent with the above utility function.

Corollary 5 *In a Duffie and Kan (1996) general equilibrium model with a log utility function, and with output and money supply processes described by assumptions A.5' and A.6', respectively, the equilibrium specification for the instantaneous nominal spot interest rate is given by:*

$$r(t) = f + \underline{G}' \cdot \underline{X}(t), \tag{68}$$

with

$$f = \rho + \pi - \left(\underline{\chi^2}\right)' \cdot \underline{\alpha},$$

and

$$\underline{G} = \underline{\phi} - \beta \cdot \underline{\chi^2}.$$

Proof. Equation (68) is simply the limit of expression (56) as $\gamma \to 0$. ■

Similarly, the risk-neutral process for $\underline{X}(t)$ that is consistent with assumption A.12" follows from Theorem 4.

Corollary 6 *In a Duffie and Kan (1996) general equilibrium model with a log utility function, and with output and money supply processes described by assumptions A.5' and A.6', respectively:*

1. *The risk-neutral process followed by the state variables under the equivalent martingale measure $\mathcal{Q}$ is equal to*

$$d\underline{X}(t) = [a \cdot \underline{X}(t) + \underline{b}]\, dt + \Sigma \cdot \sqrt{V^D(t)} \cdot d\underline{W}^{\mathcal{Q}}(t),$$

if and only if the stochastic process followed by the state variables under the original probability measure $\mathcal{P}$ is assumed to be given by:

$$d\underline{X}(t) = \left[\bar{a} \cdot \underline{X}(t) + \underline{\bar{b}}\right] dt + \Sigma \cdot \sqrt{V^D(t)} \cdot d\underline{W}^{\mathcal{P}}(t), \tag{69}$$

where

$$\bar{a} = a + \Sigma \cdot \Phi^D \cdot \beta',$$

and

$$\underline{\bar{b}} = \underline{b} + \Sigma \cdot \Phi^D \cdot \underline{\alpha}.$$

with $\Phi^D = diag\{\chi_1, \ldots, \chi_n\}$.

2. $d\underline{W}^{\mathcal{Q}}(t) = \underline{\Lambda}[\underline{X}(t)]\, dt + d\underline{W}^{\mathcal{P}}(t)$, *with*

$$\underline{\Lambda}[\underline{X}(t)] = \sqrt{V^D(t)} \cdot \underline{\chi}. \tag{70}$$

Proof. Corollary 6 is obtained from Theorem 4 by taking the limit of expressions (61) and (62), as γ tends to zero. ■

Now, Equations (68) and (69) completely specify a simpler but more restrictive $(P, \underline{\nu}, \underline{\Lambda}, \sigma)$ compatible term structure model, under a log utility function. Such specification embodies as special cases several existing equilibrium term structure models, such as Cox, Ingersoll, and Ross (1985b) and Longstaff and Schwartz (1992a), which were also derived under the restrictive type of preferences implied by the log utility function. Moreover, Equation (70) is equivalent to the market prices of risk' specification estimated by Dai and Singleton (1998, Equation 5), using the simulated method of moments, and considered by Lund (1997, Equation 25), through a linear Kalman filter implemented by QML estimation.

7 Conclusion

This article was intended to bring two main contributions to the existing literature. Firstly, in Theorems 1 and 2 new equilibrium specifications are given both for the *nominal* short-term interest rate and for the expected excess *nominal* return on a financial contingent claim, in the general context of a one-country monetary economy. Secondly, Theorems 3 and 4 propose a general equilibrium Duffie and Kan (1996) model specification, under the original probability measure $\mathcal{P}$, that is compatible with the original model' formulation under the equivalent martingale measure $\mathcal{Q}$, and that is based on more realistic assumptions about preferences than those implied by the usual Bernoulli logarithmic utility function (since a power utility function was used). In other words, our $(P, \underline{v}, \underline{\Lambda}, \sigma)$ model is a very general term structure model, not only because it is the most general in the class of the multifactor affine time-homogeneous interest rate models, but also because it relies on general assumptions about preferences and nests, as special cases, other specifications previously presented in the literature for the vector of market prices of interest rate risk. For empirical purposes, the proposed $(P, \underline{v}, \underline{\Lambda}, \sigma)$ specification is useful since it enables the econometric estimation of the Duffie and Kan (1996) model' parameters from a time-series of values for the state variables or from a panel-data of market observables.

A Appendix: Proof of Equation (28)

Imposing the two market clearing conditions -$\underline{\omega_S}' \cdot \underline{1} = 1$ and $\underline{\omega_F} = \underline{0}$- to (22), a second version for the HJB equation is obtained:

$$\max_{\left(\underline{\omega_S}, C\right)} \phi\left(\underline{\omega_S}, C; v, \underline{x}, t\right) = \max_{\left(\underline{\omega_S}, C\right)} \left\{u\left(C, t\right) + \left(L^{\underline{\omega_S}, C} J\right)\left(v, \underline{x}, t\right)\right\} = 0, \tag{71}$$

where

$$\begin{aligned} \left(L^{\underline{\omega_S}, C} J\right)\left(v, \underline{x}, t\right) \equiv\ & J_t + \left[v\underline{\omega_S}' \cdot \underline{\mu_S}\left(t\right) - p\left(t\right) C\right] J_v + J_{\underline{x}'} \cdot \left(\bar{a} \cdot \underline{x} + \underline{\bar{b}}\right) \\ & + \frac{1}{2} tr\left[J_{\underline{x}\,\underline{x}'} \cdot \Sigma \cdot V^D\left(t\right) \cdot \Sigma'\right] + \frac{1}{2} v^2 J_{vv} \underline{\omega_S}' \cdot E\left(t\right) \cdot E\left(t\right)' \cdot \underline{\omega_S} \\ & + v\underline{\omega_S}' \cdot E\left(t\right) \cdot \sqrt{V^D\left(t\right)} \cdot \Sigma' \cdot J_{v\underline{x}}, \end{aligned}$$

with $\underline{\omega_S} \geq \underline{0}$, $C \geq 0$, and subject to $J(v, \underline{x}, T) = 0$. Similarly, conditions (23) to (27) can be rewritten as:

$$\phi_C = u_C(t) - p(t) J_v \leq 0, \tag{72}$$

$$[u_C(t) - p(t) J_v] C = 0, \tag{73}$$

$$\phi_{\underline{\omega_S}} = v J_v \underline{\mu_S}(t) + v^2 J_{vv} E(t) \cdot E(t)' \cdot \underline{\omega_S} + v E(t) \cdot \sqrt{V^D(t)} \cdot \Sigma' \cdot J_{v\underline{x}} \leq \underline{0}, \tag{74}$$

$$\underline{\omega_S}' \cdot \phi_{\underline{\omega_S}} = 0, \tag{75}$$

and

$$\underline{\omega_S}' \cdot \underline{1} = 1. \tag{76}$$

Following Cox, Ingersoll, and Ross (1985a) and considering the Kuhn-Tucker Theorem, conditions (74), (75), and (76) can be rewritten as a quadratic programming problem:

$$\max_{\underline{\omega_S}} \left\{ \underline{\omega_S}' \cdot \left[v J_v \underline{\mu_S}(t) + \frac{1}{2} v^2 J_{vv} E(t) \cdot E(t)' \cdot \underline{\omega_S} + v E(t) \cdot \sqrt{V^D(t)} \cdot \Sigma' \cdot J_{v\underline{x}} \right] \right\}, \tag{77}$$

subject to $\underline{\omega_S}' \cdot \underline{1} = 1$ and $\underline{\omega_S} \geq \underline{0}$. Moreover, using l as a Lagrange multiplier, problem (77) is also equivalent to

$$\begin{aligned} \max_{(\underline{\omega_S}, l)} \quad & \left\{ \underline{\omega_S}' \cdot \left[v J_v \underline{\mu_S}(t) + \frac{1}{2} v^2 J_{vv} E(t) \cdot E(t)' \cdot \underline{\omega_S} + v E(t) \cdot \sqrt{V^D(t)} \cdot \Sigma' \cdot J_{v\underline{x}} \right] \right. \\ & \left. - l \left(\underline{\omega_S}' \cdot \underline{1} - 1 \right) \right\}, \end{aligned} \tag{78}$$

subject to $\underline{\omega_S} \geq \underline{0}$. The corresponding Kuhn-Tucker conditions are given by

$$v J_v^* \underline{\mu_S}(t) + v^2 J_{vv}^* E(t) \cdot E(t)' \cdot \underline{\omega_S}^* + v E(t) \cdot \sqrt{V^D(t)} \cdot \Sigma' \cdot J_{v\underline{x}}^* - l \underline{1} \leq \underline{0}, \tag{79}$$

and

$$\begin{aligned} 0 \quad = \quad & v J_v^* \left(\underline{\omega_S}^* \right)' \cdot \underline{\mu_S}(t) + v^2 J_{vv}^* \left(\underline{\omega_S}^* \right)' \cdot E(t) \cdot E(t)' \cdot \underline{\omega_S}^* \\ & + v \left(\underline{\omega_S}^* \right)' \cdot E(t) \cdot \sqrt{V^D(t)} \cdot \Sigma' \cdot J_{v\underline{x}}^* - l \left(\underline{\omega_S}^* \right)' \cdot \underline{1}, \end{aligned} \tag{80}$$

where $\underline{\omega_S}^*$ denotes the optimal value of $\underline{\omega_S}$, and J^* represents the indirect utility function obtained at $\underline{\omega_S} = \underline{\omega_S}^*$.

On the other hand, if it is assumed that $\underline{\omega_F} = \underline{0}$ without considering, for the moment, that $\underline{\omega_S}' \cdot \underline{1} = 1$, instead of restrictions (74), (75), and (76), one would have to deal with the following two conditions:

$$\begin{aligned} \phi_{\underline{\omega_S}} &= vJ_v \left[\underline{\mu_S}(t) - r(t)\underline{1}\right] + v^2 J_{vv} E(t) \cdot E(t)' \cdot \underline{\omega_S} \\ &\quad + vE(t) \cdot \sqrt{V^D(t)} \cdot \Sigma' \cdot J_{v\underline{x}} \\ &\leq \underline{0}, \end{aligned} \tag{81}$$

and

$$\underline{\omega_S}' \cdot \phi_{\underline{\omega_S}} = 0. \tag{82}$$

But, these last two restrictions are equivalent to another quadratic programming problem:

$$\begin{aligned} \max_{\underline{\omega_S}} \quad & \left\{ \underline{\omega_S}' \cdot \left[vJ_v \left(\underline{\mu_S}(t) - r(t)\underline{1} \right) + \frac{1}{2} v^2 J_{vv} E(t) \cdot E(t)' \cdot \underline{\omega_S} \right.\right. \\ & \left.\left. + vE(t) \cdot \sqrt{V^D(t)} \cdot \Sigma' \cdot J_{v\underline{x}} \right] \right\}, \end{aligned} \tag{83}$$

subject to $\underline{\omega_S} \geq \underline{0}$, with the associated Kuhn-Tucker conditions given by

$$\begin{aligned} \underline{0} &\geq v^2 J_{vv}^{**} E(t) \cdot E(t)' \cdot \underline{\omega_S}^{**} + vE(t) \cdot \sqrt{V^D(t)} \cdot \Sigma' \cdot J_{v\underline{x}}^{**} \\ &\quad + vJ_v^{**} \underline{\mu_S}(t) - vJ_v^{**} r(t) \underline{1}, \end{aligned} \tag{84}$$

and

$$\begin{aligned} 0 &= vJ_v^{**} \left(\underline{\omega_S}^{**}\right)' \cdot \underline{\mu_S}(t) + v^2 J_{vv}^{**} \left(\underline{\omega_S}^{**}\right)' \cdot E(t) \cdot E(t)' \cdot \underline{\omega_S}^{**} \\ &\quad + v \left(\underline{\omega_S}^{**}\right)' \cdot E(t) \cdot \sqrt{V^D(t)} \cdot \Sigma' \cdot J_{v\underline{x}}^{**} - vJ_v^{**} r(t) \left(\underline{\omega_S}^{**}\right)' \cdot \underline{1}, \end{aligned} \tag{85}$$

where $\underline{\omega_S}^{**}$ denotes the new optimal value of $\underline{\omega_S}$, and J^{**} represents the indirect utility function obtained at $\underline{\omega_S} = \underline{\omega_S}^{**}$.

Comparing (79)-(80) with (84)-(85), it follows that if $J^{**} = J^*$ and $vJ_v^{**} r(t) = l$, then $\underline{\omega_S}^{**} = \underline{\omega_S}^*$. Therefore

$$r(t) = \frac{l}{vJ_v}. \tag{86}$$

Solving (80) for l, and since $\left(\underline{\omega_S}\right)' \cdot \underline{1} = 1$,

$$\begin{aligned} l &= vJ_v \underline{\omega_S}' \cdot \underline{\mu_S}(t) + v^2 J_{vv} \underline{\omega_S}' \cdot E(t) \cdot E(t)' \cdot \underline{\omega_S} \\ &\quad + v\underline{\omega_S}' \cdot E(t) \cdot \sqrt{V^D(t)} \cdot \Sigma' \cdot J_{v\underline{x}}. \end{aligned} \tag{87}$$

Finally, combining (86) and (87), Equation (28) follows immediately. ■

B Appendix: Proof of Equation (29)

Considering that in equilibrium all wealth is invested in the physical production processes, the budget constraint (20) is given by

$$dv = \left[v\underline{\omega_S}' \cdot \underline{\mu_S}(t) - p(t)C(t)\right]dt + v\underline{\omega_S}' \cdot E(t) \cdot d\underline{W}^{\mathcal{P}}(t), \tag{88}$$

and

$$\begin{aligned}(LJ)(v,\underline{x},t) \equiv\; & J_t + \left[v\underline{\omega_S}' \cdot \underline{\mu_S}(t) - p(t)C(t)\right]J_v + J_{\underline{x}'} \cdot \left(\bar{a} \cdot \underline{x} + \bar{\underline{b}}\right) \\ & + \frac{1}{2}v^2 J_{vv}\underline{\omega_S}' \cdot E(t) \cdot E(t)' \cdot \underline{\omega_S} + \frac{1}{2}tr\left[J_{\underline{x}\,\underline{x}'} \cdot \Sigma \cdot V^D(t) \cdot \Sigma'\right] \\ & + v\underline{\omega_S}' \cdot E(t) \cdot \sqrt{V^D(t)} \cdot \Sigma' \cdot J_{v\underline{x}}.\end{aligned}$$

Applying Itô's lemma to $J_v(v,\underline{x},t)$,

$$dJ_v = \mu_{J_v}(t)\,dt + \left[vJ_{vv}\underline{\omega_S}' \cdot E(t) + \left(J_{v\underline{x}}\right)' \cdot \Sigma \cdot \sqrt{V^D(t)}\right] \cdot d\underline{W}^{\mathcal{P}}(t), \tag{89}$$

where

$$\begin{aligned}\mu_{J_v}(t) =\; & (LJ_v)(v,\underline{x},t) \\ =\; & J_{vt} + \left[v\underline{\omega_S}' \cdot \underline{\mu_S}(t) - p(t)C(t)\right]J_{vv} + J_{v\underline{x}'} \cdot \left(\bar{a} \cdot \underline{x} + \bar{\underline{b}}\right) \\ & + \frac{1}{2}v^2 J_{vvv}\underline{\omega_S}' \cdot E(t) \cdot E(t)' \cdot \underline{\omega_S} \\ & + \frac{1}{2}tr\left[J_{v\underline{x}\,\underline{x}'} \cdot \Sigma \cdot V^D(t) \cdot \Sigma'\right] \\ & + v\underline{\omega_S}' \cdot E(t) \cdot \sqrt{V^D(t)} \cdot \Sigma' \cdot J_{vv\underline{x}}.\end{aligned} \tag{90}$$

But, in equilibrium $\frac{\partial \phi}{\partial v} = 0$, that is

$$\begin{aligned}0 =\; & J_{tv} + \underline{\omega_S}' \cdot \underline{\mu_S}(t)J_v + \left[v\underline{\omega_S}' \cdot \underline{\mu_S}(t) - p(t)C(t)\right]J_{vv} \\ & + J_{\underline{x}'v} \cdot \left(\bar{a} \cdot \underline{x} + \bar{\underline{b}}\right) + vJ_{vv}\underline{\omega_S}' \cdot E(t) \cdot E(t)' \cdot \underline{\omega_S} \\ & + \frac{1}{2}v^2 J_{vvv}\underline{\omega_S}' \cdot E(t) \cdot E(t)' \cdot \underline{\omega_S} \\ & + \frac{1}{2}tr\left[J_{\underline{x}\,\underline{x}'v} \cdot \Sigma \cdot V^D(t) \cdot \Sigma'\right]\end{aligned} \tag{91}$$

$$+\underline{\omega_S}' \cdot E(t) \cdot \sqrt{V^D(t)} \cdot \Sigma' \cdot J_{v\underline{x}}$$
$$+v\underline{\omega_S}' \cdot E(t) \cdot \sqrt{V^D(t)} \cdot \Sigma' \cdot J_{v\underline{x}v}.$$

Hence, combining Equations (90) and (91),

$$\mu_{J_v}(t) = -\underline{\omega_S}' \cdot \underline{\mu_S}(t) J_v - vJ_{vv}\underline{\omega_S}' \cdot E(t) \cdot E(t)' \cdot \underline{\omega_S} - \underline{\omega_S}' \cdot E(t) \cdot \sqrt{V^D(t)} \cdot \Sigma' \cdot J_{v\underline{x}}.$$

Consequently,

$$\begin{aligned} -E\left(\frac{dJ_v}{J_v}\right)\frac{1}{dt} &= -\frac{\mu_{J_v}}{J_v} \\ &= \underline{\omega_S}' \cdot \underline{\mu_S}(t) + v\left(\frac{J_{vv}}{J_v}\right)\underline{\omega_S}' \cdot E(t) \cdot E(t)' \cdot \underline{\omega_S} \\ &\quad + \underline{\omega_S}' \cdot E(t) \cdot \sqrt{V^D(t)} \cdot \Sigma' \cdot \left(\frac{J_{v\underline{x}}}{J_v}\right). \end{aligned}$$

Comparing the above equality with Equation (28), yields Equation (29). ■

C Appendix: An alternative proof of Theorem 1

Following Bakshi and Chen (1997b, pages 818-819), formula (39) will be derived by assuming from the beginning a pure exchange economy. Moreover, instead of working in continuous time, the current Appendix will start by considering time intervals of length Δt, and later it will take $\Delta t \to 0$. Although the same closed-form solution for $r(t)$ will be now obtained in a much simpler fashion, such derivation does not provide any intuition towards the qualitative results obtained in Section 4; and, it is also subject to a discretization error of small magnitude.

As usual, the time-t price of a default-free (unit face value) pure discount bond with maturity at time $(t + \Delta t)$ will be denominated by $P(t, t + \Delta t)$. When $\Delta t \to 0$, $r(t)$ is the yield-to-maturity of such zero coupon bond, and therefore

$$P(t, t + \Delta t) = e^{-r(t)\Delta t}.$$

On the other hand, it is well known that, in equilibrium, the loss of marginal utility of current consumption implied by the purchase of a pure discount bond at time t must be equal to the gain of expected marginal utility of future consumption implied by the obtention of a monetary unit at time $(t + \Delta t)$. And, since $P(t, t + \Delta t)$ monetary units at time t correspond to $\frac{P(t,t+\Delta t)}{p(t)}$ units of real consumption, one monetary unit received at time $(t+\Delta t)$ is equivalent to $\frac{1}{p(t+\Delta t)}$ units of real consumption, and it is assumed that $C(t) = q(t)$, then

$$\frac{P(t, t+\Delta t)}{p(t)} u_q[q(t), t] = E_t \left\{ \frac{1}{p(t+\Delta t)} u_q[q(t+\Delta t), t+\Delta t] \right\}.$$

Combining the last two equations, and considering that $u(q,t) = e^{-\rho t} U(q)$,[14]

$$1 + [\rho - r(t)] \Delta t \approx E_t \left\{ \frac{U_q[q(t+\Delta t)]}{U_q[q(t)]} \frac{p(t)}{p(t+\Delta t)} \right\}. \tag{92}$$

Taking the Taylor series expansion of $\frac{1}{p(t+\Delta t)}$ at $p(t)$,

$$\frac{1}{p(t+\Delta t)} = \frac{1}{p(t)} - \frac{1}{p(t)^2} \Delta p(t) + \frac{1}{2} \frac{2}{p(t)^3} [\Delta p(t)]^2 + O\left(\Delta t^{\frac{3}{2}}\right),$$

where $O\left(\Delta t^{\frac{3}{2}}\right)$ is a linear function of $\Delta t^{\frac{3}{2}}$ and higher-order terms, which are negligible. Multiplying both sides of the above equation by $p(t)$,

$$\frac{p(t)}{p(t+\Delta t)} = 1 - \frac{\Delta p(t)}{p(t)} + \left[\frac{\Delta p(t)}{p(t)} \right]^2 + O\left(\Delta t^{\frac{3}{2}}\right). \tag{93}$$

Taking the Taylor series expansion of $p(t+\Delta t) = \frac{M(t+\Delta t)}{q(t+\Delta t)}$ at $(M(t), q(t))$,

$$\begin{aligned} p(t+\Delta t) &= p(t) + \frac{\Delta M(t)}{q(t)} - \frac{M(t)}{q(t)^2} \Delta q(t) + \frac{M(t)}{q(t)^3} [\Delta q(t)]^2 \\ &\quad - \frac{1}{q(t)^2} \Delta q(t) \Delta M(t) + O\left(\Delta t^{\frac{3}{2}}\right), \end{aligned}$$

that is

$$\frac{\Delta p(t)}{p(t)} = \frac{\Delta M(t)}{M(t)} - \frac{\Delta q(t)}{q(t)} + \left[\frac{\Delta q(t)}{q(t)} \right]^2 - \frac{\Delta q(t)}{q(t)} \frac{\Delta M(t)}{M(t)} + O\left(\Delta t^{\frac{3}{2}}\right). \tag{94}$$

Substituting (94) into (93), and taking $\Delta t \to 0$,

$$\begin{aligned} \frac{p(t)}{p(t+\Delta t)} &= 1 - \frac{dM(t)}{M(t)} + \frac{dq(t)}{q(t)} - \underline{\sigma_q}(t)' \cdot \underline{\sigma_q}(t) dt + \sigma_{q,M}(t) dt \\ &\quad + \underline{\sigma_M}(t)' \cdot \underline{\sigma_M}(t) dt + \underline{\sigma_q}(t)' \cdot \underline{\sigma_q}(t) dt - 2\sigma_{q,M}(t) dt + O\left(dt^{\frac{3}{2}}\right), \end{aligned}$$

i.e.

$$\frac{p(t)}{p(t+\Delta t)} = 1 + \left[\underline{\sigma_M}(t)' \cdot \underline{\sigma_M}(t) - \sigma_{q,M}(t) \right] dt - \frac{dM(t)}{M(t)} + \frac{dq(t)}{q(t)} + O\left(dt^{\frac{3}{2}}\right). \tag{95}$$

Again, taking the Taylor series of $U_q[q(t+\Delta t)]$ at $q(t)$,

$$U_q[q(t+\Delta t)] = U_q[q(t)] + U_{qq}[q(t)] \Delta q(t) + \frac{1}{2} U_{qqq}[q(t)] [\Delta q(t)]^2 + O\left(\Delta t^{\frac{3}{2}}\right),$$

and making $\Delta t \to 0$,

$$\begin{aligned} \frac{U_q\left[q\left(t+dt\right)\right]}{U_q\left[q\left(t\right)\right]} &= 1+\frac{U_{qq}\left[q\left(t\right)\right]}{U_q\left[q\left(t\right)\right]}dq\left(t\right) \\ &+\frac{1}{2}\frac{U_{qqq}\left[q\left(t\right)\right]}{U_q\left[q\left(t\right)\right]}\underline{\sigma_q}\left(t\right)'\cdot\underline{\sigma_q}\left(t\right)q\left(t\right)^2dt+O\left(dt^{\frac{3}{2}}\right). \end{aligned} \tag{96}$$

Substituting (96) and (95) into (92), making $\Delta t \to 0$, and considering only time-dependencies,

$$\begin{aligned} 1+\left[\rho-r\left(t\right)\right]dt &= E_t\left\{1+\frac{U_{qq}\left(t\right)}{U_q\left(t\right)}dq\left(t\right)+\frac{1}{2}\frac{U_{qqq}\left(t\right)}{U_q\left(t\right)}\underline{\sigma_q}\left(t\right)'\cdot\underline{\sigma_q}\left(t\right)q\left(t\right)^2dt\right. \\ &+\left[\underline{\sigma_M}\left(t\right)'\cdot\underline{\sigma_M}\left(t\right)-\sigma_{q,M}\left(t\right)\right]dt-\frac{dM\left(t\right)}{M\left(t\right)}+\frac{dq\left(t\right)}{q\left(t\right)} \\ &\left.-\frac{q\left(t\right)U_{qq}\left(t\right)}{U_q\left(t\right)}\sigma_{q,M}\left(t\right)dt+\frac{q\left(t\right)U_{qq}\left(t\right)}{U_q\left(t\right)}\underline{\sigma_q}\left(t\right)'\cdot\underline{\sigma_q}\left(t\right)dt+O\left(dt^{\frac{3}{2}}\right)\right\}. \end{aligned}$$

Taking expectations,

$$\begin{aligned} \left[\rho-r\left(t\right)\right]dt &= \frac{U_{qq}\left(t\right)}{U_q\left(t\right)}\mu_q\left(t\right)q\left(t\right)dt+\frac{1}{2}\frac{U_{qqq}\left(t\right)}{U_q\left(t\right)}\underline{\sigma_q}\left(t\right)'\cdot\underline{\sigma_q}\left(t\right)q\left(t\right)^2dt \\ &+\left[\underline{\sigma_M}\left(t\right)'\cdot\underline{\sigma_M}\left(t\right)-\sigma_{q,M}\left(t\right)\right]dt-\mu_M\left(t\right)dt+\mu_q\left(t\right)dt \\ &-\frac{q\left(t\right)U_{qq}\left(t\right)}{U_q\left(t\right)}\sigma_{q,M}\left(t\right)dt+\frac{q\left(t\right)U_{qq}\left(t\right)}{U_q\left(t\right)}\underline{\sigma_q}\left(t\right)'\cdot\underline{\sigma_q}\left(t\right)dt. \end{aligned}$$

Dividing both sides by dt, and because $\frac{U_{qq}(t)}{U_q(t)}=\frac{u_{qq}(t)}{u_q(t)}$ as well as $\frac{U_{qqq}(t)}{U_q(t)}=\frac{u_{qqq}(t)}{u_q(t)}$, one obtains exactly the same result as in Equation (39). ■

D Appendix: An alternative proof of Theorem 2

Once again, a pure exchange economy is assumed, and time intervals of length Δt are initially considered.

In equilibrium, the factor risk premiums on any contingent claim $F\left(t\right)$ must obey to the following Euler equation:

$$E_t\left\{\frac{U_q\left[q\left(t+\Delta t\right),t+\Delta t\right]}{U_q\left[q\left(t\right),t\right]}\frac{p\left(t\right)}{p\left(t+\Delta t\right)}\left[\frac{\Delta F\left(t\right)}{F\left(t\right)}-r\left(t\right)\Delta t\right]\right\}=0.$$

Making $\Delta t \to 0$, and substituting $\frac{U_q[q(t+\Delta t),t+\Delta t]}{U_q[q(t),t]}$ by Equation (96) as well as $\frac{p(t)}{p(t+\Delta t)}$ by Equation (95), yields:

$$\begin{aligned} 0 &= E_t\left\{\left[1+\frac{U_{qq}\left(t\right)}{U_q\left(t\right)}dq\left(t\right)+\frac{1}{2}\frac{U_{qqq}\left(t\right)}{U_q\left(t\right)}\underline{\sigma_q}\left(t\right)'\cdot\underline{\sigma_q}\left(t\right)q\left(t\right)^2dt\right.\right. \\ &+\left[\underline{\sigma_M}\left(t\right)'\cdot\underline{\sigma_M}\left(t\right)-\sigma_{q,M}\left(t\right)\right]dt-\frac{dM\left(t\right)}{M\left(t\right)}+\frac{dq\left(t\right)}{q\left(t\right)}-\frac{q\left(t\right)U_{qq}\left(t\right)}{U_q\left(t\right)}\sigma_{q,M}\left(t\right)dt \\ &\left.\left.+\frac{q\left(t\right)U_{qq}\left(t\right)}{U_q\left(t\right)}\underline{\sigma_q}\left(t\right)'\cdot\underline{\sigma_q}\left(t\right)dt+O\left(dt^{\frac{3}{2}}\right)\right]\left[\left(\frac{dF\left(t\right)}{F\left(t\right)}-r\left(t\right)dt\right)\right]\right\}. \end{aligned}$$

Multiplying both members inside the expectation operator, and taking expectations,

$$\begin{aligned}0 &= \mu_F(t)\,dt - r(t)\,dt + \frac{U_{qq}(t)}{U_q(t)} COV\left[\frac{dF(t)}{F(t)}, dq(t)\right] dt \\ &\quad - COV\left[\frac{dF(t)}{F(t)}, \frac{dM(t)}{M(t)}\right] dt + COV\left[\frac{dF(t)}{F(t)}, \frac{dq(t)}{q(t)}\right] dt.\end{aligned}$$

Finally, dividing both sides of the above equation by dt and rearranging terms, Equation (45) follows. ■

E Appendix: An alternative derivation of $\underline{\Lambda}\left[\underline{X}(t)\right]$ under a power utility function

Since $\underline{\mu}\left[\underline{X}(t)\right] = \underline{v}\left[\underline{X}(t)\right] - \sigma\left[\underline{X}(t)\right] \cdot \underline{\Lambda}\left[\underline{X}(t)\right]$, because $\sigma\left[\underline{X}(t)\right]$ is assumed to be invertible, and using Equations (5) and (10), then

$$\underline{\Lambda}\left[\underline{X}(t)\right] = \left[\Sigma \cdot \sqrt{V^D(t)}\right]^{-1} \cdot \left\{\left[\bar{a} \cdot \underline{X}(t) + \underline{\bar{b}}\right] - \left[a \cdot \underline{X}(t) + \underline{b}\right]\right\}.$$

Attending to Equation (61),

$$\underline{\Lambda}\left[\underline{X}(t)\right] = \left[\sqrt{V^D(t)}\right]^{-1} \cdot \Sigma^{-1} \cdot \left[\Sigma \cdot \Omega^D \cdot \beta' \cdot \underline{X}(t) + \Sigma \cdot \Omega^D \cdot \underline{\alpha}\right].$$

Finally, since $\Sigma \cdot \Omega^D \cdot \beta' \cdot \underline{X}(t) + \Sigma \cdot \Omega^D \cdot \underline{\alpha} = \Sigma \cdot V^D(t) \cdot \left(\underline{\chi} - \gamma\underline{\varphi}\right)$, then

$$\underline{\Lambda}\left[\underline{X}(t)\right] = \sqrt{V^D(t)} \cdot \left(\underline{\chi} - \gamma\underline{\varphi}\right),$$

as expected.

Notes

1. An affine form corresponds to a constant plus a linear function.

2. And stated, for instance, in Lamberton and Lapeyre (1996, Theorem 3.5.5).

3. Meaning that the relative prices of all assets with respect to the numeraire given by a "money market account" are $\mathcal{Q}$-martingales. The time-t value of such "savings account", $\delta(t)$, corresponds to the compounded value of one monetary unit continuously reinvested, from time 0 to time t, at the short-term interest rate $r(\cdot)$:
$$\delta(t) = \exp\left[\int_0^t r(s)\, ds\right].$$

4. As Duffie and Kan (1996, page 381) say: "...the yields are affine if, and essentially only if, the drift and diffusion functions of the stochastic differential equation for the factors are also affine". This result is equivalent to proposition 4 of Brown and Schaefer (1994), derived in the context of one-factor affine models.

5. As defined in Arnold (1992, definition 2.6.1). Its relation with the infinitesimal generator of $\underline{X}(t)$, $\mathcal{A}$, is the following:
$$\mathcal{A} = \frac{\partial}{\partial t} + \mathcal{D}.$$

6. Or substituting $Y(\underline{x}, t)$, in PDE (7), by Equation (3), subject to the boundary condition $P(T,T) = 1$.

7. $J(v, \underline{x}, t)$ represents the indirect utility function of the representative agent, expressed in terms of the *nominal* wealth. Although the direct utility function is assumed to be state-independent, we can not be sure in saying the same about the indirect utility function because $r(t)$ changes stochastically.

8. In order to simplify the notation, subscripts will be used hereafter to represent derivatives.

9. Both types of economy can be made compatible through the definition of both $\underline{\mu_S}(q, M, \underline{S}, \underline{X}, t)$ and $E(q, M, \underline{S}, \underline{X}, t)$ in such a way that the production economy generates an endogenous consumption

process identical to the exogenously specified output process. See Heston (1988, footnote 9) or Bakshi and Chen (1997a, footnote 5).

10. $-\frac{C(t)u_{CC}(t)}{u_C(t)} = 1-\gamma$, i.e. constant relative risk aversion (CRRA) is being assumed.

11. Which, accordingly to Ingersoll (1987, Chapter 1, Equation 51), can be summarized as

$$u(q,t) = e^{-\rho t}\frac{1-\gamma}{\gamma}\left(\frac{aq}{1-\gamma}+b\right)^{\gamma}, \ b>0, \ a,\gamma \in \Re.$$

12. Combining Equations (45) and (47).

13. An alternative derivation is provided in Appendix E.

14. The above mentioned discretization error arises from the fact that $\exp\{[\rho - r(t)]\,\Delta t\}$ is only equal to $\{1+[\rho - r(t)]\,\Delta t\}$, or equivalently, terms of order higher than Δt can only be ignored, when $\Delta t \to 0$. In other words, such approximation ignores the difference between continuous and discrete compounding.

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Table 1: Parameters' restrictions needed to fit some term structure models into the Duffie and Kan (1996) general specification

	Vasicek (1977)	Cox, Ingersoll, and Ross (1985b)	Longstaff and Schwartz (1992a)	Langetieg (1980)	Chen and Scott (1995a)
n	1	1	2		
f	0	0	0		0
$\underline{G}$	1	1			$\underline{1}$
a			diagonal		diagonal
$\underline{b}$					
Σ			I_2		diagonal
$\underline{\alpha}$	1	0	$\underline{0}$		$\underline{0}$
β	0	1	I_2	O_n	I_n

O_n and I_n denote $n \times n$ null and identity matrices, respectively.

$\underline{\alpha} \in \Re^n$ is a vector with α_i as its i^{th}-component.

$\beta \in \Re^{n \times n}$ is a matrix whose i^{th}-column is given by vector $\underline{\beta_i}$.

All the other variables are defined according to the terminology of Duffie and Kan (1996).