

MULTI-FACTOR VALUATION OF FLOATING RANGE NOTES

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ABSTRACT. Under a one-factor Gaussian HJM model, Turnbull (1995) as well as Navatte and Quittard-Pinon (1999) provide explicit pricing solutions for *range notes* contracts.

The present paper generalizes such closed-form solutions for the context of a multi-factor Gaussian HJM framework.

KEY WORDS: Gaussian HJM multi-factor models, Change of probability measure, Bivariate normal distribution, Interest rate digital options, Range notes.

1. INTRODUCTION

The main purpose of this paper is to derive exact analytical valuation formulas for *floating range notes* in the context of a multi-factor Gaussian Heath, Jarrow, and Morton (1992) term structure model. As an accessory result, exact and explicit pricing solutions are also provided for *fixed range notes*.

A *floating range note*, as described in Turnbull (1995), pays a floating coupon rate at the end of each compounding period, based on the value of some reference interest rate (e.g. 3-month US Libor) in the beginning of each compounding period. However, and unlike a standard floating-rate note, the coupon also depends on (or is proportional to) the number of days that the reference interest rate lies inside a *corridor*¹, during each compounding period. This last feature complicates the valuation of these interest rate structured products, since it induces a path-dependency on the reference interest rate process. For a *fixed range note*, each coupon is equal to a pre-specified (or fixed) annual interest rate divided by the number of days in a year² and multiplied by the number of days, of the compounding period, where some reference interest rate lies inside a *corridor*.

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¹That is between some pre-specified lower (r_l) and upper (r_u) interest rate levels, which can vary across different compounding periods (or even daily).

²Computed under some daycount convention.

Using a one-factor Gaussian HJM model, Turnbull (1995) has priced explicitly each coupon of a floating range note as a portfolio of *range-contingent payoff options*³ plus -see Turnbull (1995, equation 20)- an extra term, which only involves the univariate normal distribution function. Under the same framework but using the change of numeraire technique, Navatte and Quittard-Pinon (1999) have rewritten each coupon of a floating range note as a portfolio of *double delayed digital options* plus -see Navatte and Quittard-Pinon (1999, equation 14)- the same extra term, only involving the univariate normal distribution function. In both cases, such extra term arises from the correlation between the values of the reference interest rate at two different dates: on the last coupon date and on each business day until the next coupon date. When moving towards a multi-factor framework, this paper shows that the same structure will be obtained for the price of each coupon, being the only difference the fact that the previously mentioned extra term will have to be expressed as an integral over a bivariate normal density function. Nevertheless, such extra term will still be obtained in closed-form, under a multi-factor Gaussian HJM framework.

Therefore, the present paper can be thought as a simple and straightforward extension that follows from the suggestion put forward by Navatte and Quittard-Pinon (1999, page 440): “Another challenge would be to consider the valuation of range notes with two state variables governing the multifactor corridor”. This paper derives exact and explicit solutions for the same interest rate structured products but under a Gaussian HJM model driven by arbitrarily many stochastic factors, and using the well-known change of numeraire/measure technique.

The proposed extension towards a multi-factor formulation is important because -as noticed, for instance, by Rebonato (1998, page 70)- it enhances the term structure model’ calibration to the interest rates covariance matrix “observed” in the market, which along with the term structure of interest rates will ultimately determine the price of the range notes under analysis. In fact, in order to price and hedge the exotic interest rate products under analysis (range notes) consistently with the market prices of related plain-vanilla interest rate options (such as caps and/or European swaptions), it is essential to use an interest rate model that satisfies two requisites. First, the term structure model must be analytically tractable, in the sense that it should provide expedite valuation formulae for both the exotic interest rate product and for the underlying plain-vanilla options. Such requisite is clearly satisfied by the simple Gaussian framework used in this paper. Second, the calibration of the HJM model under consideration must provide a good fit to the term structures of interest rates⁴, of volatilities and of correlations observed in the market. This second goal can only be better

³One for each business day of the compounding period.

⁴This goal is naturally achieved by construction.

achieved by considering a multi-factor formulation. Besides the unrealistic perfect correlation assumption amongst all interest rates intrinsic to any single-factor term structure model, such framework would clearly encompass too few parameters to fit satisfactorily the market prices of all relevant underlying plain-vanilla options. Moreover, the proposed multi-factor formulation can also accommodate the Principal Components Analysis' usual prescription of three stylized factors: *level*, *slope* and *curvature* (see, for instance, Litterman and Scheinkman (1991)).

The analytical tractability provided by the proposed multi-factor Gaussian -but not necessarily Markovian or time-homogeneous- Heath, Jarrow, and Morton (1992) term structure model is obtained at the expense of an important theoretical limitation: interest rates are assumed to be normally distributed, and therefore can attain negative values with positive probability.⁵ However, the extension to stochastic volatility structures is outside the scope of the present paper, whose contribution is, nevertheless, the assumption of arbitrarily many stochastic risk factors.

Next sections are organized as follows. Section 2 describes the most relevant probabilistic features of the multi-factor Gaussian HJM model that will be used hereafter. Section 3 provides closed-form solutions for the interest rate digital options that will be used to price range notes. Then, section 4 prices fixed range notes and generalizes, under a multi-factor formulation, the analytical solutions obtained by Turnbull (1995) for floating range notes. Finally, section 5 concludes.

2. MULTI-FACTOR GAUSSIAN HJM

2.1. Model description

Hereafter, \mathcal{Q} will denote the martingale probability measure obtained when the “money market account” is taken as the numeraire of the economy underlying the model under analysis.⁶ In such underlying stochastic intertemporal economy there exists a trading interval $\mathcal{T} = [t_0, \tau]$, for some fixed time $\tau > t_0$, and uncertainty is represented by a probability space $(\Omega, \mathcal{F}, \mathcal{Q})$, where all the information accruing to all the agents in the economy is described by a filtration $(\mathcal{F}_t)_{t \in \mathcal{T}}$ satisfying the usual conditions: namely, \mathcal{F}_{t_0} is assumed to be almost trivial, and $\mathcal{F}_\tau = \mathcal{F}$.

⁵Rogers (1996) identifies the potential pricing implications of such model “deficiency”, even for small (risk-neutral) probabilities of attaining negative interest rates.

⁶Meaning that the relative prices of all assets with respect to the numeraire given by a “money market account” are \mathcal{Q} -martingales.

It is further assumed that there exists an arbitrage-free⁷ and frictionless market for pure discount bond prices, which are considered to be perfectly divisible and to evolve through time according to the following stochastic differential equation:

$$(2.1) \quad \frac{dP(t, T)}{P(t, T)} = r(t) dt + \sigma(t, T)' \cdot dW^{\mathcal{Q}}(t),$$

where $P(t, T)$ represents the time- t price of a (unit face value) zero coupon bond expiring at time T , for all $T \in [t_0, \tau]$ and $t \in [t_0, T]$, $r(t)$ is the time- t instantaneous spot rate, which can be defined by continuity as

$$r(t) := \lim_{T \rightarrow t} \left[-\frac{\ln P(t, T)}{T - t} \right],$$

\cdot denotes the inner product in \mathfrak{R}^n , $:=$ means equal by definition, and $W^{\mathcal{Q}}(t) \in \mathfrak{R}^n$ is a n -dimensional standard Brownian motion, initialized at zero and generating the augmented, right continuous and complete filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq t_0\}$.

The n -dimensional adapted volatility function $\sigma(\cdot, T) : [t_0, T] \rightarrow \mathfrak{R}^n$ is assumed to satisfy the usual mild measurability and integrability requirements -as stated, for instance, in Lamberton and Lapeyre (1996, theorem 3.5.5)- as well as the boundary condition⁸ $\sigma(u, u) = \underline{0} \in \mathfrak{R}^n, \forall u \in [t_0, T]$. Moreover, for reasons of analytical tractability, that is in order to obtain lognormally distributed pure discount bond prices, such volatility function is assumed to be deterministic.

Equation (2.1), equipped with the assumption of a deterministic volatility function, represents the *Gaussian* interest rate term structure model that will be used to derive exact and analytical pricing solutions for range notes.

2.2. Probability densities for log pure discount bond prices

Next two propositions offer the probabilistic tools needed to price interest rate digital options.

PROPOSITION 2.1. *Let \mathcal{Q}^{T_a} be the equivalent martingale measure obtained when the numeraire is taken to be a pure discount bond with expiry date at time T_a . Under the Gaussian HJM specification (2.1), and conditional on \mathcal{F}_{t_0} , the random variable $\ln P(T_a, T_b)$, with $t_0 \leq T_a \leq T_b$, possesses a univariate normal*

⁷Following Harrison and Pliska (1981), the absence of arbitrage is implied by the existence of the equivalent martingale measure \mathcal{Q} .

⁸Consistently with the “pull-to-par” phenomena.

distribution with mean $\ln \left[\frac{P(t_0, T_b)}{P(t_0, T_a)} \right] - \frac{1}{2}g(t_0, T_a, T_b)$ and standard deviation $\sqrt{g(t_0, T_a, T_b)}$, that is⁹

$$(2.2) \quad \mathcal{Q}^{T_a} [\ln P(T_a, T_b) \in dx] = \phi \left\{ x; \ln \left[\frac{P(t_0, T_b)}{P(t_0, T_a)} \right] - \frac{1}{2}g(t_0, T_a, T_b), \sqrt{g(t_0, T_a, T_b)} \right\} dx,$$

where¹⁰

$$(2.3) \quad g(t_0, T_a, T_b) := \int_{t_0}^{T_a} \|\sigma(s, T_b) - \sigma(s, T_a)\|^2 ds.$$

PROOF. Using equation (2.1) and applying Itô's lemma to $\ln P(t, T)$,

$$(2.4) \quad \ln P(t, T) = \ln P(t_0, T) + \int_{t_0}^t \left[r(s) - \frac{1}{2} \sigma(s, T)' \cdot \sigma(s, T) \right] ds + \int_{t_0}^t \sigma(s, T)' \cdot dW^{\mathcal{Q}}(s).$$

Hence, the forward pure discount bond price process, under measure \mathcal{Q} , is given by

$$(2.5) \quad \ln \left[\frac{P(t, T_b)}{P(t, T_a)} \right] = \ln \left[\frac{P(t_0, T_b)}{P(t_0, T_a)} \right] - \frac{1}{2} \int_{t_0}^t \left[\|\sigma(s, T_b)\|^2 - \|\sigma(s, T_a)\|^2 \right] ds + \int_{t_0}^t [\sigma(s, T_b) - \sigma(s, T_a)]' \cdot dW^{\mathcal{Q}}(s).$$

Following Geman, Karoui, and Rochet (1995), the T_a -forward martingale measure will be assumed to exist and to be defined on the same measurable space (Ω, \mathcal{F}) as measure \mathcal{Q} , through the following *Radon-Nikodym derivative*

$$(2.6) \quad \frac{d\mathcal{Q}^{T_a}}{d\mathcal{Q}} \Big|_{\mathcal{F}_t} := \frac{P(t, T_a)}{P(t_0, T_a)} \frac{\beta(t_0)}{\beta(t)},$$

for $t \leq T_a$, and where $\beta(t)$ represents the time- t value of the “money market account”, i.e. the compounded value of one monetary unit continuously reinvested, from time t_0 to time t , at the short-term interest rate:

$$(2.7) \quad \beta(t) := \exp \left[\int_{t_0}^t r(s) ds \right].$$

Combining equations (2.4) and (2.7), definition (2.6) can be restated as

$$(2.8) \quad \frac{d\mathcal{Q}^{T_a}}{d\mathcal{Q}} \Big|_{\mathcal{F}_t} = \exp \left[\int_{t_0}^t \sigma(s, T_a)' \cdot dW^{\mathcal{Q}}(s) - \frac{1}{2} \int_{t_0}^t \|\sigma(s, T_a)\|^2 ds \right].$$

Consequently, if measure \mathcal{Q}^{T_a} exists,¹¹ then *Girsanov's theorem* implies that

$$(2.9) \quad dW^{\mathcal{Q}^{T_a}}(t) = dW^{\mathcal{Q}}(t) - \sigma(t, T_a) dt$$

is also a vector of standard Brownian motion increments in \mathfrak{R}^n (with the same standard filtration as $dW^{\mathcal{Q}}(t)$),

for $t \leq T_a$.

⁹Hereafter, $\phi(X; \mu, \sigma) := (2\pi\sigma^2)^{-\frac{1}{2}} \exp \left[-\frac{1}{2} \frac{(X-\mu)^2}{\sigma^2} \right]$ represents the probability density function of a normally distributed univariate random variable X , with mean μ and standard deviation σ . $\mathcal{Q}^T(\omega | \mathcal{F}_t)$ denotes the probability of event ω , conditional on \mathcal{F}_t , and computed under the equivalent martingale measure \mathcal{Q}^T . Since \mathcal{F}_{t_0} is assumed to be trivial, the expectation on the left-hand-side of equation (2.2) has been taken unconditionally.

¹⁰ $\|\cdot\|$ denotes the Euclidean norm in \mathfrak{R}^n .

¹¹That is as long as $\sigma(\cdot, T_a) \in L_2[t_0, t]$, in the sense that, with probability one, $\int_{t_0}^t \|\sigma(s, T_a)\|^2 ds < \infty$. Since function σ is deterministic, the corresponding *Novikov's condition* is automatically satisfied.

The forward pure discount bond price process, under measure \mathcal{Q}^{T_a} , can now be obtained through equations (2.5) and (2.9):

$$(2.10) \quad \ln \left[\frac{P(t, T_b)}{P(t, T_a)} \right] = \ln \left[\frac{P(t_0, T_b)}{P(t_0, T_a)} \right] - \frac{1}{2} \int_{t_0}^t \|\sigma(s, T_b) - \sigma(s, T_a)\|^2 ds \\ + \int_{t_0}^t [\sigma(s, T_b) - \sigma(s, T_a)]' \cdot dW^{\mathcal{Q}^{T_a}}(s).$$

Replacing t by T_a , and using definition (2.3),

$$(2.11) \quad \ln P(T_a, T_b) = \ln \left[\frac{P(t_0, T_b)}{P(t_0, T_a)} \right] - \frac{1}{2} g(t_0, T_a, T_b) + \int_{t_0}^{T_a} [\sigma(s, T_b) - \sigma(s, T_a)]' \cdot dW^{\mathcal{Q}^{T_a}}(s).$$

Finally, considering that $g(t_0, T_a, T_b) < \infty$, and using, for instance, Arnold (1992, corollary 4.5.6), it follows that the stochastic integral contained in the last equality is normally distributed with mean zero and variance $g(t_0, T_a, T_b)$. ■

REMARK 2.1. *Proposition 2.1 will be used to price standard European digital options, where the payoff occurs at the same time that the underlying reference interest rate is revealed.*

PROPOSITION 2.2. *Let \mathcal{Q}^{T_c} be the equivalent martingale measure obtained when the numeraire is taken to be a pure discount bond with expiry date at time $T_c (\geq t_0)$. Under the Gaussian HJM specification (2.1), and conditional on \mathcal{F}_{t_0} , the random variable $\ln P(T_a, T_b)$, with $t_0 \leq T_a \leq T_b$, possesses a univariate normal distribution with mean $\ln \left[\frac{P(t_0, T_b)}{P(t_0, T_a)} \right] - \frac{1}{2} g(t_0, T_a, T_b) + l(t_0, T_a, T_b, T_c)$ and standard deviation $\sqrt{g(t_0, T_a, T_b)}$, that is*

$$(2.12) \quad \mathcal{Q}^{T_c} [\ln P(T_a, T_b) \in dx] \\ = \phi \left\{ x; \ln \left[\frac{P(t_0, T_b)}{P(t_0, T_a)} \right] - \frac{1}{2} g(t_0, T_a, T_b) + l(t_0, T_a, T_b, T_c), \sqrt{g(t_0, T_a, T_b)} \right\} dx,$$

where

$$(2.13) \quad l(t_0, T_a, T_b, T_c) := \int_{t_0}^{T_a} [\sigma(s, T_b) - \sigma(s, T_a)]' \cdot [\sigma(s, T_c) - \sigma(s, T_a)] ds.$$

PROOF. Applying the same steps as in deriving identity (2.9), it is easy to show that the existence assumption of the forward measure \mathcal{Q}^{T_c} implies that

$$(2.14) \quad dW^{\mathcal{Q}^{T_c}}(t) = dW^{\mathcal{Q}}(t) - \sigma(t, T_c) dt$$

is a n -dimensional vector of Brownian motion increments. Combining equalities (2.9) and (2.14),

$$(2.15) \quad dW^{\mathcal{Q}^{T_a}}(t) = dW^{\mathcal{Q}^{T_c}}(t) + [\sigma(t, T_c) - \sigma(t, T_a)] dt.$$

Equation (2.15) can now be used to rewrite the process (2.11) under measure \mathcal{Q}^{T_c} :

$$(2.16) \quad \ln P(T_a, T_b) = \ln \left[\frac{P(t_0, T_b)}{P(t_0, T_a)} \right] - \frac{1}{2} g(t_0, T_a, T_b) + l(t_0, T_a, T_b, T_c) \\ + \int_{t_0}^{T_a} [\sigma(s, T_b) - \sigma(s, T_a)]' \cdot dW^{\mathcal{Q}^{T_c}}(s),$$

where the deterministic function $l(t_0, T_a, T_b, T_c)$ is defined by equation (2.13). Again, Arnold (1992, corollary 4.5.6) implies that the stochastic integral contained in equation (2.16) is normally distributed with mean zero and variance $g(t_0, T_a, T_b)$. ■

REMARK 2.2. *Proposition 2.2 will be used to price European delayed digital options, where there is a time-gap between the occurrence of the terminal payoff (posterior) and the settlement of the underlying reference interest rate (anterior).*

3. INTEREST RATE DIGITAL OPTIONS

Since the fair price of any range note will be shown to include series of interest rate digital options, the present section prices explicitly such basic contingent claims following closely Turnbull (1995, sections 2, 3 and 4). For all the digital options to be considered, the underlying will be some time- $t \in \mathcal{T}$ nominal¹² spot rate for a given compounding period $\delta \in \mathfrak{R}$, which can be defined as

$$(3.1) \quad r_n(t, t + \delta) := \frac{1}{\delta} \left[\frac{1}{P(t, t + \delta)} - 1 \right].$$

Standard European interest rate digital calls (puts) possess a terminal payoff equal to one if and only if, at maturity, the underlying interest rate is above (bellow) a pre-specified strike rate. More formally,

DEFINITION 3.1. *The time- T price of a standard European digital call (put) option on the nominal spot rate $r_n(T, T + \delta)$, with a strike rate equal to r_k , and maturity at time T is equal to*

$$(3.2) \quad SD(\theta)_T[r_n(T, T + \delta); r_k; T] := \begin{cases} 1 & \iff \theta r_n(T, T + \delta) > \theta r_k \\ 0 & \iff \theta r_n(T, T + \delta) \leq \theta r_k \end{cases},$$

where $\theta = 1$ for a digital call or $\theta = -1$ for a digital put.

PROPOSITION 3.1. *Under the Gaussian HJM model (2.1), the time- t ($\leq T$) price of a standard European digital call (put) option on the nominal spot rate $r_n(T, T + \delta)$, with a strike rate equal to r_k , and maturity at time T is equal to*

$$(3.3) \quad SD(\theta)_t[r_n(T, T + \delta); r_k; T] = P(t, T) \Phi[\theta d(r_k)],$$

¹²I.e. discretely (as opposed to continuously) compounded interest rate.

with

$$(3.4) \quad d(r) := \frac{\ln \left[\frac{P(t,T)}{P(t,T+\delta)(1+\delta r)} \right] + \frac{1}{2}g(t,T,T+\delta)}{\sqrt{g(t,T,T+\delta)}},$$

$\theta = 1$ for a digital call, $\theta = -1$ for a digital put, and where Φ represents the cumulative density function of the univariate standard normal distribution.

PROOF. Assuming that measure \mathcal{Q}^T exists, and following Harrison and Pliska (1981), the relative price, with respect to the T -maturity zero-coupon bond, of any attainable contingent claim that settles at time T will be a \mathcal{Q}^T -martingale. Therefore,¹³

$$(3.5) \quad SD(\theta)_t[r_n(T, T+\delta); r_k; T] = P(t, T) E_{\mathcal{Q}^T} \left[1_{\{\theta r_n(T, T+\delta) > \theta r_k\}} \middle| \mathcal{F}_t \right].$$

Because the expectation of an indicator function can be written as a probability, and using definition (3.1),

$$(3.6) \quad SD(\theta)_t[r_n(T, T+\delta); r_k; T] = P(t, T) \mathcal{Q}^T \left[\theta \ln P(T, T+\delta) < \theta \ln(1 + \delta r_k)^{-1} \middle| \mathcal{F}_t \right].$$

Finally, the probability contained in the right-hand-side of equation (3.6) can be computed explicitly through proposition 2.1, yielding the analytical solution (3.3). ■

Standard European interest rate range digital options provide a terminal payoff equal to one if and only if, at maturity, the underlying interest rate is inside a pre-specified corridor. That is,

DEFINITION 3.2. *The time- T price of a standard European range digital option on the nominal spot rate $r_n(T, T+\delta)$, with a lower rate bound equal to r_l , an upper rate bound equal to r_u ($> r_l$), and maturity at time T is equal to*

$$(3.7) \quad SRD_T[r_n(T, T+\delta); r_l; r_u; T] := \begin{cases} 1 & \iff r_n(T, T+\delta) \in [r_l; r_u] \\ 0 & \iff r_n(T, T+\delta) \notin [r_l; r_u] \end{cases}.$$

PROPOSITION 3.2. *Under the Gaussian HJM model (2.1), the time- t ($\leq T$) price of a standard European range digital option on the nominal spot rate $r_n(T, T+\delta)$, with a lower rate bound equal to r_l , an upper rate bound equal to r_u ($> r_l$), and maturity at time T is equal to*

$$(3.8) \quad SRD_t[r_n(T, T+\delta); r_l; r_u; T] = P(t, T) \{ \Phi[d(r_l)] - \Phi[d(r_u)] \}.$$

¹³ $E_{\mathcal{Q}^T}(X | \mathcal{F}_t)$ denotes the expected value of the random variable X , conditional on \mathcal{F}_t , and computed under the equivalent martingale measure \mathcal{Q}^T . Next formulae also use an indicator function, defined as:

$$1_{\{\omega \in \Omega\}} = \begin{cases} 1 & \iff \omega \in \Omega \\ 0 & \iff \omega \notin \Omega \end{cases}.$$

PROOF. Assuming again that measure \mathcal{Q}^T exists,

$$\begin{aligned}
& SRD_t[r_n(T, T + \delta); r_l; r_u; T] \\
&= P(t, T) E_{\mathcal{Q}^T} [1_{\{r_l \leq r_n(T, T + \delta) \leq r_u\}} | \mathcal{F}_t] \\
(3.9) \quad &= P(t, T) \mathcal{Q}^T \left[\ln(1 + \delta r_u)^{-1} \leq \ln P(T, T + \delta) \leq \ln(1 + \delta r_l)^{-1} \mid \mathcal{F}_t \right].
\end{aligned}$$

Applying proposition 2.1, equation (3.8) is obtained. ■

REMARK 3.1. *Combining propositions 3.1 and 3.2,*

$$(3.10) \quad SRD_t[r_n(T, T + \delta); r_l; r_u; T] = SD(1)_t[r_n(T, T + \delta); r_l; T] - SD(1)_t[r_n(T, T + \delta); r_u; T],$$

or

$$(3.11) \quad SRD_t[r_n(T, T + \delta); r_l; r_u; T] = SD(-1)_t[r_n(T, T + \delta); r_u; T] - SD(-1)_t[r_n(T, T + \delta); r_l; T].$$

See, for instance, Turnbull (1995, equation 6).

All the previous option contracts assume that the terminal payoff occurs on the same date as the fixing of the underlying interest rate. However, for range notes there is typically a time gap between the knowledge of the underlying reference rate (on each coupon day) and the occurrence of the terminal payoff (on the next coupon date). Such feature will be captured by the following *delayed* interest rate digital option contracts.

DEFINITION 3.3. *The time- T_1 price of a delayed European digital call (put) option on the nominal spot rate $r_n(T, T + \delta)$, with a strike rate equal to r_k , and maturity at time T_1 ($\geq T$) is equal to*

$$(3.12) \quad DD(\theta)_{T_1}[r_n(T, T + \delta); r_k; T_1] := \begin{cases} 1 & \iff \theta r_n(T, T + \delta) > \theta r_k \\ 0 & \iff \theta r_n(T, T + \delta) \leq \theta r_k \end{cases},$$

where $\theta = 1$ for a digital call or $\theta = -1$ for a digital put.

PROPOSITION 3.3. *Under the Gaussian HJM model (2.1), the time- t ($\leq T$) price of a delayed European digital call (put) option on the nominal spot rate $r_n(T, T + \delta)$, with a strike rate equal to r_k , and maturity at time T_1 ($\geq T$) is equal to*

$$(3.13) \quad DD(\theta)_t[r_n(T, T + \delta); r_k; T_1] = P(t, T_1) \Phi[\theta h(r_k)],$$

with

$$(3.14) \quad h(r) := \frac{\ln \left[\frac{P(t, T)}{P(t, T + \delta)(1 + \delta r)} \right] + \frac{1}{2}g(t, T, T + \delta) - l(t, T, T + \delta, T_1)}{\sqrt{g(t, T, T + \delta)}},$$

and where $\theta = 1$ for a digital call or $\theta = -1$ for a digital put.

PROOF. Assuming that measure \mathcal{Q}^{T_1} exists,

$$\begin{aligned}
 DD(\theta)_t[r_n(T, T + \delta); r_k; T_1] &= P(t, T_1) E_{\mathcal{Q}^{T_1}} [1_{\{\theta r_n(T, T + \delta) > \theta r_k\}} | \mathcal{F}_t] \\
 (3.15) \qquad \qquad \qquad &= P(t, T_1) \mathcal{Q}^{T_1} \left[\theta \ln P(T, T + \delta) < \theta \ln(1 + \delta r_k)^{-1} \mid \mathcal{F}_t \right].
 \end{aligned}$$

Applying proposition 2.2 (with $t_0 = t$, $T_a = T$, $T_b = T + \delta$, and $T_c = T_1$), proposition 3.3 follows. ■

DEFINITION 3.4. *The time- T_1 price of a delayed European range digital option on the nominal spot rate $r_n(T, T + \delta)$, with a lower rate bound equal to r_l , an upper rate bound equal to r_u ($> r_l$), and maturity at time T_1 ($\geq T$) is equal to*

$$(3.16) \qquad \qquad \qquad DRD_{T_1}[r_n(T, T + \delta); r_l; r_u; T_1] := \begin{cases} 1 & \Leftarrow r_n(T, T + \delta) \in [r_l; r_u] \\ 0 & \Leftarrow r_n(T, T + \delta) \notin [r_l; r_u] \end{cases}.$$

PROPOSITION 3.4. *Under the Gaussian HJM model (2.1), the time- t ($\leq T$) price of a delayed European range digital option on the nominal spot rate $r_n(T, T + \delta)$, with a lower rate bound equal to r_l , an upper rate bound equal to r_u ($> r_l$), and maturity at time T_1 ($\geq T$) is equal to*

$$(3.17) \qquad \qquad \qquad DRD_t[r_n(T, T + \delta); r_l; r_u; T_1] = P(t, T_1) \{\Phi[h(r_l)] - \Phi[h(r_u)]\}.$$

PROOF. Assuming again that measure \mathcal{Q}^{T_1} exists,

$$\begin{aligned}
 &DRD_t[r_n(T, T + \delta); r_l; r_u; T_1] \\
 &= P(t, T_1) E_{\mathcal{Q}^{T_1}} [1_{\{r_l \leq r_n(T, T + \delta) \leq r_u\}} | \mathcal{F}_t] \\
 (3.18) \qquad \qquad \qquad &= P(t, T_1) \mathcal{Q}^{T_1} \left[\ln(1 + \delta r_u)^{-1} \leq \ln P(T, T + \delta) \leq \ln(1 + \delta r_l)^{-1} \mid \mathcal{F}_t \right].
 \end{aligned}$$

Applying proposition 2.2, equation (3.17) follows. ■

REMARK 3.2. *As usual,*

$$(3.19) \qquad DRD_t[r_n(T, T + \delta); r_l; r_u; T_1] = DD(1)_t[r_n(T, T + \delta); r_l; T_1] - DD(1)_t[r_n(T, T + \delta); r_u; T_1],$$

or

$$\begin{aligned}
 (3.20) \qquad \qquad \qquad DRD_t[r_n(T, T + \delta); r_l; r_u; T_1] &= \quad DD(-1)_t[r_n(T, T + \delta); r_u; T_1] \\
 &\quad - DD(-1)_t[r_n(T, T + \delta); r_l; T_1].
 \end{aligned}$$

See, for instance, Turnbull (1995, equation 13) or Navatte and Quittard-Pinon (1999, equation 10).

4. RANGE NOTES

The main purpose of this section is to generalize the closed-form solutions provided by Turnbull (1995, equation 23) and Navatte and Quittard-Pinon (1999, equation 15), for floating range notes, to the context of the multi-factor formulation under analysis. However, the approach adopted by the previous authors to price the next coupon of a floating range note will be first applied to the valuation of fixed range notes.

In what follows, let t denote the valuation date, and consider a bond with bullet redemption, with its last coupon date at time T_0 ($\leq t$), and with N future coupons v_{j+1} , with payment dates at times T_{j+1} ($> t$), $j = 0, \dots, N - 1$. Denote by n_j (δ_j) the number of days (years) between times T_j and T_{j+1} (based on some daycount convention). For the current coupon period, set $n_0 = n_0^- + n_0^+$, where n_0^- (n_0^+) represents the number of days between times T_0 and t (t and T_1). Finally, define by $T_{j,i}$ the date corresponding to i days after time T_j , and take as $\delta_{j,i}$ the length of the compounding period (in years) that starts at time $T_{j,i}$. Diagram 4.1 summarizes the cash flows associated with the generic coupon-bearing bond under analysis.

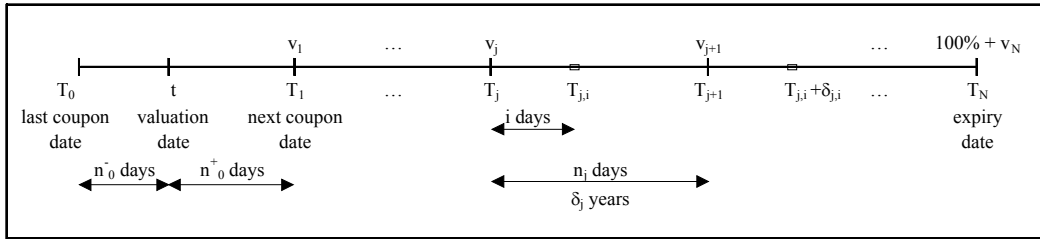


FIGURE 4.1. Pattern of future cash flows for a generic range note

4.1. Fixed range notes

As for a floating range note, each coupon value of a fixed range note depends on the number of days (during the coupon period) that a reference interest rate lies inside an interest rate *corridor*. Nevertheless, the valuation of a fixed range note is less involved because all coupon rates are pre-specified when the bond is issued.

DEFINITION 4.1. For a fixed range note, the value of the $(j + 1)^{th}$ coupon, at time T_{j+1} , is equal to

$$(4.1) \quad v_{j+1}(T_{j+1}) := C_j \frac{H(T_j, T_{j+1})}{D_j},$$

where C_j represents the (annual) coupon rate for the $(j + 1)^{\text{th}}$ compounding period,¹⁴ D_j is the number of days in a year for the $(j + 1)^{\text{th}}$ compounding period,¹⁵ and

$$(4.2) \quad H(T_j, T_{j+1}) := \sum_{i=1}^{n_j} 1_{\{r_l(T_{j,i}) \leq r_n(T_{j,i}, T_{j,i} + \delta_{j,i}) \leq r_u(T_{j,i})\}}$$

denotes the number of days, in the $(j + 1)^{\text{th}}$ compounding period, that the reference interest rate lies inside a pre-specified range,¹⁶ which is equal to $[r_l(T_{j,i}); r_u(T_{j,i})]$ for the i^{th} day of the $(j + 1)^{\text{th}}$ compounding period (with $r_l(T_{j,i}) < r_u(T_{j,i})$).

PROPOSITION 4.1. *Under the Gaussian HJM model (2.1), the time- t price of a fixed range note with bullet redemption, with its last coupon paid at time T_0 ($\leq t$), and with N future coupons v_{j+1} specified by definition 4.1 and paid at times T_{j+1} ($> t$), $j = 0, \dots, N - 1$, is equal to*

$$(4.3) \quad \text{FiRN}(t) = P(t, T_N) + v_1(t) + \sum_{j=1}^{N-1} v_{j+1}(t),$$

with

$$(4.4) \quad v_1(t) = \frac{C_0}{D_0} \left\{ P(t, T_1) H(T_0, t) + \sum_{i=1}^{n_0^+} DRD_t \left[r_n(T_{0, n_0^- + i}, T_{0, n_0^- + i} + \delta_{0, n_0^- + i}); r_l(T_{0, n_0^- + i}); r_u(T_{0, n_0^- + i}); T_1 \right] \right\},$$

and

$$(4.5) \quad v_{j+1}(t) = \frac{C_j}{D_j} \sum_{i=1}^{n_j} DRD_t [r_n(T_{j,i}, T_{j,i} + \delta_{j,i}); r_l(T_{j,i}); r_u(T_{j,i}); T_{j+1}],$$

and where C_j , D_j , and n_j represent, respectively, the coupon rate, the number of days in a year, and the number of days for the $(j + 1)^{\text{th}}$ compounding period, $[r_l(T_{j,i}); r_u(T_{j,i})]$ defines the corridor for the i^{th} day of the $(j + 1)^{\text{th}}$ coupon period, n_0^- and n_0^+ represent the number of days in the time interval $[T_0, t]$ and $[t, T_1]$, respectively, and

$$(4.6) \quad H(T_0, t) := \sum_{i=1}^{n_0^-} 1_{\{r_l(T_{0,i}) \leq r_n(T_{0,i}, T_{0,i} + \delta_{0,i}) \leq r_u(T_{0,i})\}}.$$

PROOF. The first term on the right-hand-side of equation (4.3) is simply the present value of the face value to be received on the expiry date.

Concerning the next coupon, using equation (4.1), and assuming that measure \mathcal{Q}^{T_1} exists,

$$(4.7) \quad v_1(t) = \frac{C_0}{D_0} P(t, T_1) E_{\mathcal{Q}^{T_1}} [H(T_0, T_1) | \mathcal{F}_t].$$

¹⁴Known at time t .

¹⁵Based on some daycount convention.

¹⁶Possibly, but not usually, different for each day.

Because $H(T_0, T_1)$ is partially known at time t , denoting by $H(T_0, t)$ ($\leq n_0^-$)-as defined by equation (4.6)-the number of accrued days in the current coupon period that have had the reference interest rate inside the interest rate range, and considering definition (4.2), equation (4.7) becomes

$$\begin{aligned} v_1(t) &= \frac{C_0}{D_0} P(t, T_1) \left\{ H(T_0, t) + E_{\mathcal{Q}^{T_1}} \left[\sum_{i=1}^{n_0^+} 1_{\left\{ r_l(T_{0, n_0^-+i}) \leq r_n(T_{0, n_0^-+i}, T_{0, n_0^-+i} + \delta_{0, n_0^-+i}) \leq r_u(T_{0, n_0^-+i}) \right\}} \middle| \mathcal{F}_t \right] \right\} \\ &= \frac{C_0}{D_0} \{ P(t, T_1) H(T_0, t) \\ &\quad + \sum_{i=1}^{n_0^+} P(t, T_1) E_{\mathcal{Q}^{T_1}} \left[1_{\left\{ r_l(T_{0, n_0^-+i}) \leq r_n(T_{0, n_0^-+i}, T_{0, n_0^-+i} + \delta_{0, n_0^-+i}) \leq r_u(T_{0, n_0^-+i}) \right\}} \middle| \mathcal{F}_t \right] \}. \end{aligned}$$

Using definition 3.4, each term inside the last summation sign can be written as the time t value of a delayed European range digital option on the nominal spot rate $r_n(T_{0, n_0^-+i}, T_{0, n_0^-+i} + \delta_{0, n_0^-+i})$, with a lower rate bound equal to $r_l(T_{0, n_0^-+i})$, an upper rate bound equal to $r_u(T_{0, n_0^-+i})$, and maturity at time T_1 , which yields equation (4.4).

Finally, but similarly, for any other future coupon v_{j+1} ($j = 1, \dots, N-1$), and assuming that measure $\mathcal{Q}^{T_{j+1}}$ exists, equations (4.1) and (4.2) yield

$$\begin{aligned} v_{j+1}(t) &= \frac{C_j}{D_j} P(t, T_{j+1}) E_{\mathcal{Q}^{T_{j+1}}} \left[\sum_{i=1}^{n_j} 1_{\left\{ r_l(T_{j,i}) \leq r_n(T_{j,i}, T_{j,i} + \delta_{j,i}) \leq r_u(T_{j,i}) \right\}} \middle| \mathcal{F}_t \right] \\ (4.8) \quad &= \frac{C_j}{D_j} \sum_{i=1}^{n_j} P(t, T_{j+1}) \mathcal{Q}^{T_{j+1}} [r_l(T_{j,i}) \leq r_n(T_{j,i}, T_{j,i} + \delta_{j,i}) \leq r_u(T_{j,i}) | \mathcal{F}_t]. \end{aligned}$$

Using again definition 3.4, the last summation can be written as a portfolio of delayed European range digital options, and equation (4.5) follows. ■

4.2. Floating range notes

The pricing of the next coupon of a floating range note is similar to the valuation of fixed range notes because the coupon rate is known since the last coupon date. However, in order to price any of the following coupon payments it will be necessary to consider the correlation between the unknown coupon rate and the reference interest rate on each day of the coupon period. And, although a multi-factor term structure model is in use, it will be shown that such future cash flows can be priced explicitly. For that purpose, the following result will be used.

LEMMA 4.2. For $b, \rho \in \mathfrak{R}$,

$$(4.9) \quad \int_{-\infty}^{\infty} (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}w^2\right) \Phi\left(\frac{b-\rho w}{\sqrt{1-\rho^2}}\right) dw = \Phi(b).$$

PROOF. For $a \in \mathfrak{R}$, and using, for instance, Geske (1979, page 80):

$$(4.10) \quad \int_{-\infty}^a (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}w^2\right) \Phi\left(\frac{b-\rho w}{\sqrt{1-\rho^2}}\right) dw = M(a; b; \rho),$$

where $M(a; b; \rho)$ represents the cumulative probability, in a standardized bivariate normal distribution, that the first variable is less than a and the second variable is less than b , when the coefficient of correlation between the variables is ρ . Similarly, applying the change of variables $y = -w$ to equation (4.10),

$$(4.11) \quad \int_a^{\infty} (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}y^2\right) \Phi\left(\frac{b-\rho y}{\sqrt{1-\rho^2}}\right) dy = M(-a; b; -\rho).$$

Combining identities (4.10) and (4.11), and using the following relation

$$(4.12) \quad M(a; b; \rho) + M(-a; b; -\rho) = \Phi(b),$$

which can be found, for instance, in Drezner (1978, equation 8), equation (4.9) is obtained. ■

DEFINITION 4.2. For a floating range note, the value of the $(j+1)^{\text{th}}$ coupon, at time T_{j+1} , is equal to

$$(4.13) \quad v_{j+1}(T_{j+1}) := \frac{r_n(T_j, T_j + \delta_j) + s_j}{D_j} H(T_j, T_{j+1}),$$

where s_j represents the spread over the reference interest rate paid by the bond during the $(j+1)^{\text{th}}$ compounding period,¹⁷ D_j is the number of days in a year for the $(j+1)^{\text{th}}$ compounding period, and $H(T_j, T_{j+1})$ is defined as in equation (4.2).

Next proposition contains the main result of this paper.

PROPOSITION 4.3. Under the Gaussian HJM model (2.1), the time- t price of a floating range note with bullet redemption, with its last coupon paid at time T_0 ($\leq t$), and with N future coupons v_{j+1} specified by definition 4.2 and paid at times T_{j+1} ($> t$), $j = 0, \dots, N-1$, is equal to

$$(4.14) \quad FLRN(t) = P(t, T_N) + v_1(t) + \sum_{j=1}^{N-1} v_{j+1}(t),$$

with

$$(4.15) \quad v_1(t) = \frac{r_n(T_0, T_0 + \delta_0) + s_0}{D_0} \left\{ P(t, T_1) H(T_0, t) + \sum_{i=1}^{n_0^+} DRD_t \left[r_n(T_{0, n_0^- + i}, T_{0, n_0^- + i} + \delta_{0, n_0^- + i}); r_l(T_{0, n_0^- + i}); r_u(T_{0, n_0^- + i}); T_1 \right] \right\},$$

¹⁷Known at time t .

and

$$(4.16) \quad v_{j+1}(t) = \left(\frac{s_j}{D_j} - \frac{1}{\delta_j D_j} \right) \sum_{i=1}^{n_j} DRD_t [r_n(T_{j,i}, T_{j,i} + \delta_{j,i}); r_l(T_{j,i}); r_u(T_{j,i}); T_{j+1}] \\ + \frac{2P(t, T_{j+1})}{\delta_j D_j} \sum_{i=1}^{n_j} [\varepsilon_{j,i} (4a_{j,i} b_{j,i} - c_{j,i}^2)]^{-\frac{1}{2}} \exp \left[\frac{e_{j,i}^2}{8b_{j,i}} - \frac{f_{j,i}}{2} \right. \\ \left. + \frac{(c_{j,i} e_{j,i} - 4b_{j,i} - 2b_{j,i} d_{j,i})^2}{8b_{j,i} (4a_{j,i} b_{j,i} - c_{j,i}^2)} \right] \{ \Phi[\theta_{j,i}(r_l(T_{j,i}))] - \Phi[\theta_{j,i}(r_u(T_{j,i}))] \},$$

and where

$$(4.17) \quad \theta_{j,i}(r) := \frac{2a_{j,i} e_{j,i} - 2c_{j,i} - d_{j,i} c_{j,i}}{2\sqrt{a_{j,i}} \sqrt{4a_{j,i} b_{j,i} - c_{j,i}^2}} - \frac{\sqrt{4a_{j,i} b_{j,i} - c_{j,i}^2}}{2\sqrt{a_{j,i}}} \ln(1 + \delta_{j,i} r),$$

$$(4.18) \quad a_{j,i} := \{ [1 - \rho^2(T_j, T_{j,i})] g(t, T_j, T_{j+1}) \}^{-1},$$

$$(4.19) \quad b_{j,i} := \{ [1 - \rho^2(T_j, T_{j,i})] g(t, T_{j,i}, T_{j,i} + \delta_{j,i}) \}^{-1},$$

$$(4.20) \quad c_{j,i} := -\frac{2 [1 - \rho^2(T_j, T_{j,i})]^{-1} \rho(T_j, T_{j,i})}{\sqrt{g(t, T_j, T_{j+1}) g(t, T_{j,i}, T_{j,i} + \delta_{j,i})}},$$

$$(4.21) \quad d_{j,i} := 2 [1 - \rho^2(T_j, T_{j,i})]^{-1} \left[\frac{\rho(T_j, T_{j,i}) \mu(t, T_{j,i}, T_{j,i} + \delta_{j,i})}{\sqrt{g(t, T_j, T_{j+1}) g(t, T_{j,i}, T_{j,i} + \delta_{j,i})}} - \frac{\mu(t, T_j, T_{j+1})}{g(t, T_j, T_{j+1})} \right],$$

$$(4.22) \quad e_{j,i} := 2 [1 - \rho^2(T_j, T_{j,i})]^{-1} \left[\frac{\rho(T_j, T_{j,i}) \mu(t, T_j, T_{j+1})}{\sqrt{g(t, T_j, T_{j+1}) g(t, T_{j,i}, T_{j,i} + \delta_{j,i})}} - \frac{\mu(t, T_{j,i}, T_{j,i} + \delta_{j,i})}{g(t, T_{j,i}, T_{j,i} + \delta_{j,i})} \right],$$

$$(4.23) \quad f_{j,i} := [1 - \rho^2(T_j, T_{j,i})]^{-1} \left[\frac{\mu^2(t, T_j, T_{j+1})}{g(t, T_j, T_{j+1})} + \frac{\mu^2(t, T_{j,i}, T_{j,i} + \delta_{j,i})}{g(t, T_{j,i}, T_{j,i} + \delta_{j,i})} \right. \\ \left. - 2 \frac{\rho(T_j, T_{j,i}) \mu(t, T_j, T_{j+1}) \mu(t, T_{j,i}, T_{j,i} + \delta_{j,i})}{\sqrt{g(t, T_j, T_{j+1}) g(t, T_{j,i}, T_{j,i} + \delta_{j,i})}} \right],$$

$$(4.24) \quad \varepsilon_{j,i} := g(t, T_j, T_{j+1}) g(t, T_{j,i}, T_{j,i} + \delta_{j,i}) [1 - \rho^2(T_j, T_{j,i})],$$

$$(4.25) \quad \mu(t, T_a, T_b) := \ln \left[\frac{P(t, T_b)}{P(t, T_a)} \right] - \frac{1}{2} g(t, T_a, T_b) + l(t, T_a, T_b, T_{j+1}),$$

$$(4.26) \quad \rho(T_j, T_{j,i}) := \frac{\int_t^{T_j} [\sigma(s, T_{j+1}) - \sigma(s, T_j)]' \cdot [\sigma(s, T_{j,i} + \delta_{j,i}) - \sigma(s, T_{j,i})] ds}{\sqrt{g(t, T_j, T_{j+1}) g(t, T_{j,i}, T_{j,i} + \delta_{j,i})}},$$

s_j , D_j , and n_j (δ_j) represent, respectively, the spread over the reference interest rate, the number of days in a year, and the number of days (years) for the $(j+1)^{th}$ compounding period, $[r_l(T_{j,i}); r_u(T_{j,i})]$ defines the corridor for the i^{th} day of the $(j+1)^{th}$ coupon period, n_0^- and n_0^+ represent the number of days in the time interval $[T_0, t]$ and $[t, T_1]$, respectively, and $H(T_0, t)$ is defined by equation (4.6).

PROOF. As for equation (4.3), the first term on the right-hand-side of equation (4.14) corresponds to the face value discounted from the expiry date to the valuation date.

Concerning the next coupon, because the next coupon rate $-r_n(T_0, T_0 + \delta_0) + s_0$ is already known (since time T_0), using equation (4.13), and assuming that measure \mathcal{Q}^{T_1} exists,

$$(4.27) \quad v_1(t) = \frac{r_n(T_0, T_0 + \delta_0) + s_0}{D_0} P(t, T_1) E_{\mathcal{Q}^{T_1}} [H(T_0, T_1) | \mathcal{F}_t].$$

Because the $H(T_0, t)$ portion of $H(T_0, T_1)$ is already known at time t , and considering definition (4.2), equation (4.27) yields

$$(4.28) \quad \begin{aligned} v_1(t) &= \frac{r_n(T_0, T_0 + \delta_0) + s_0}{D_0} \{P(t, T_1) H(T_0, t) \\ &\quad + \sum_{i=1}^{n_0^+} P(t, T_1) E_{\mathcal{Q}^{T_1}} \left[\mathbb{1}_{\left\{ r_l(T_{0, n_0^- + i}) \leq r_n(T_{0, n_0^- + i}, T_{0, n_0^- + i} + \delta_{0, n_0^- + i}) \leq r_u(T_{0, n_0^- + i}) \right\}} \middle| \mathcal{F}_t \right] \}. \end{aligned}$$

Using definition 3.4, equation (4.15) follows.

For any other future coupon v_{j+1} ($j = 1, \dots, N-1$), and assuming that measure $\mathcal{Q}^{T_{j+1}}$ exists, equations (4.2) and (4.13) yield

$$(4.29) \quad \begin{aligned} v_{j+1}(t) &= P(t, T_{j+1}) E_{\mathcal{Q}^{T_{j+1}}} \left[\frac{r_n(T_j, T_j + \delta_j) + s_j}{D_j} \sum_{i=1}^{n_j} \mathbb{1}_{\{r_l(T_{j,i}) \leq r_n(T_{j,i}, T_{j,i} + \delta_{j,i}) \leq r_u(T_{j,i})\}} \middle| \mathcal{F}_t \right] \\ &= \frac{s_j}{D_j} \sum_{i=1}^{n_j} P(t, T_{j+1}) E_{\mathcal{Q}^{T_{j+1}}} \left[\mathbb{1}_{\{r_l(T_{j,i}) \leq r_n(T_{j,i}, T_{j,i} + \delta_{j,i}) \leq r_u(T_{j,i})\}} \middle| \mathcal{F}_t \right] \\ &\quad + \frac{P(t, T_{j+1})}{D_j} \sum_{i=1}^{n_j} E_{\mathcal{Q}^{T_{j+1}}} \left[r_n(T_j, T_j + \delta_j) \mathbb{1}_{\{r_l(T_{j,i}) \leq r_n(T_{j,i}, T_{j,i} + \delta_{j,i}) \leq r_u(T_{j,i})\}} \middle| \mathcal{F}_t \right]. \end{aligned}$$

While definition 3.4 implies that the first term on the right-hand-side of equation (4.29) is simply a portfolio of delayed European range digital options, its second term is more involved since it includes two different random variables (for each i): $r_n(T_j, T_j + \delta_j)$ and $r_n(T_{j,i}, T_{j,i} + \delta_{j,i})$. In order to model the joint probability density function of those random variables, the second sum in equation (4.29) will be rewritten in terms of log pure discount bond prices, using identity (3.1):

$$(4.30) \quad \begin{aligned} v_{j+1}(t) &= \left(\frac{s_j}{D_j} - \frac{1}{\delta_j D_j} \right) \sum_{i=1}^{n_j} DRD_t [r_n(T_{j,i}, T_{j,i} + \delta_{j,i}); r_l(T_{j,i}); r_u(T_{j,i}); T_{j+1}] \\ &\quad + \frac{P(t, T_{j+1})}{\delta_j D_j} \sum_{i=1}^{n_j} E_{\mathcal{Q}^{T_{j+1}}} \left[\frac{\mathbb{1}_{\{\ln(1 + \delta_{j,i} r_u(T_{j,i}))^{-1} \leq \ln P(T_{j,i}, T_{j,i} + \delta_{j,i}) \leq \ln(1 + \delta_{j,i} r_l(T_{j,i}))^{-1}\}}}{\exp(\ln P(T_j, T_j + \delta_j))} \middle| \mathcal{F}_t \right]. \end{aligned}$$

From proposition 2.2, it follows that $\ln P(T_j, T_j + \delta_j)$ possesses, conditional on \mathcal{F}_t and under the forward measure $\mathcal{Q}^{T_{j+1}}$, a univariate normal density with mean $\mu(t, T_j, T_{j+1})$, as defined by equation (4.25), and

variance $g(t, T_j, T_{j+1})$, that is

$$(4.31) \quad \mathcal{Q}^{T_{j+1}} [\ln P(T_j, T_j + \delta_j) \in dx | \mathcal{F}_t] = \phi \left\{ x; \mu(t, T_j, T_{j+1}), \sqrt{g(t, T_j, T_{j+1})} \right\} dx.$$

Similarly, proposition 2.2 also implies that $\ln P(T_{j,i}, T_{j,i} + \delta_{j,i})$ possesses, conditional on \mathcal{F}_t and under the forward measure $\mathcal{Q}^{T_{j+1}}$, a univariate normal density with mean $\mu(t, T_{j,i}, T_{j,i} + \delta_{j,i})$, as defined by equation (4.25), and variance $g(t, T_{j,i}, T_{j,i} + \delta_{j,i})$, i.e.

$$(4.32) \quad \mathcal{Q}^{T_{j+1}} [\ln P(T_{j,i}, T_{j,i} + \delta_{j,i}) \in dx | \mathcal{F}_t] = \phi \left\{ x; \mu(t, T_{j,i}, T_{j,i} + \delta_{j,i}), \sqrt{g(t, T_{j,i}, T_{j,i} + \delta_{j,i})} \right\} dx.$$

In order to show that the joint probability distribution of the random variables $\ln P(T_j, T_j + \delta_j)$ and $\ln P(T_{j,i}, T_{j,i} + \delta_{j,i})$ is a bivariate normal distribution it is still necessary to consider their correlation structure.¹⁸ Using equation (2.16) and Arnold (1992, theorem 5.1.1), the covariance between the two random variables can be defined as the following deterministic function,¹⁹

$$(4.33) \quad E_{\mathcal{Q}^{T_{j+1}}} \{ [\ln P(T_j, T_j + \delta_j) - \mu(t, T_j, T_{j+1})] [\ln P(T_{j,i}, T_{j,i} + \delta_{j,i}) - \mu(t, T_{j,i}, T_{j,i} + \delta_{j,i})] | \mathcal{F}_t \} \\ = \int_t^{\min(T_j, T_{j,i})} [\sigma(s, T_{j+1}) - \sigma(s, T_j)]' \cdot [\sigma(s, T_{j,i} + \delta_{j,i}) - \sigma(s, T_{j,i})] ds,$$

which yields equation (4.26) for the corresponding linear correlation coefficient.

The joint probability law of $\ln P(T_j, T_j + \delta_j)$ and $\ln P(T_{j,i}, T_{j,i} + \delta_{j,i})$ can be defined in terms of their joint moment generating function. For that purpose, consider a linear combination of both random variables, whose stochastic process can be obtained from equation (2.16):

$$(4.34) \quad t_a \ln P(T_j, T_j + \delta_j) + t_b \ln P(T_{j,i}, T_{j,i} + \delta_{j,i}) \\ = t_a \mu(t, T_j, T_{j+1}) + t_b \mu(t, T_{j,i}, T_{j,i} + \delta_{j,i}) + t_a \int_t^{T_j} [\sigma(s, T_{j+1}) - \sigma(s, T_j)]' \cdot dW^{\mathcal{Q}^{T_{j+1}}}(s) \\ + t_b \int_t^{T_{j,i}} [\sigma(s, T_{j,i} + \delta_{j,i}) - \sigma(s, T_{j,i})]' \cdot dW^{\mathcal{Q}^{T_{j+1}}}(s),$$

for some constants $t_a, t_b \in \mathfrak{R}$. Using, for instance, Arnold (1992, corollary 4.5.6 and theorem 5.1.1), such linear combination can be shown to also possess a univariate normal density function of the following form:

$$(4.35) \quad \mathcal{Q}^{T_{j+1}} \{ [t_a \ln P(T_j, T_j + \delta_j) + t_b \ln P(T_{j,i}, T_{j,i} + \delta_{j,i})] \in dx | \mathcal{F}_t \} \\ = \phi \left\{ x; t_a \mu(t, T_j, T_{j+1}) + t_b \mu(t, T_{j,i}, T_{j,i} + \delta_{j,i}), [t_a^2 g(t, T_j, T_{j+1}) + t_b^2 g(t, T_{j,i}, T_{j,i} + \delta_{j,i}) \right. \\ \left. + 2t_a t_b \rho(T_j, T_{j,i}) \sqrt{g(t, T_j, T_{j+1}) g(t, T_{j,i}, T_{j,i} + \delta_{j,i})}]^{\frac{1}{2}} \right\} dx,$$

¹⁸The normality of each univariate random variable is a necessary but not sufficient condition. In fact, Kowalski (1973) provides several examples of non-normal bivariate distributions for which the marginal distributions are both normal.

¹⁹ $\min(a, b)$ represents the minimum between the real numbers a and b .

where $\rho(T_j, T_{j,i})$ is defined by equation (4.26). Consequently, the moment generating function of the above defined affine function is equal to

$$(4.36) \quad \begin{aligned} & E_{\mathcal{Q}^{T_{j+1}}} \left\{ \exp [s (t_a \ln P (T_j, T_j + \delta_j) + t_b \ln P (T_{j,i}, T_{j,i} + \delta_{j,i}))] \mid \mathcal{F}_t \right\} \\ &= \exp \left\{ s [t_a \mu (t, T_j, T_{j+1}) + t_b \mu (t, T_{j,i}, T_{j,i} + \delta_{j,i})] + \frac{s^2}{2} [t_a^2 g (t, T_j, T_{j+1}) \right. \\ & \quad \left. + t_b^2 g (t, T_{j,i}, T_{j,i} + \delta_{j,i}) + 2t_a t_b \rho (T_j, T_{j,i}) \sqrt{g (t, T_j, T_{j+1}) g (t, T_{j,i}, T_{j,i} + \delta_{j,i})}] \right\}, \end{aligned}$$

for some constant $s \in \mathfrak{R}$. Taking $s = 1$, the left-hand-side of equation (4.36) can be understood as the joint moment generating function of the bivariate random variable $[\ln P (T_j, T_j + \delta_j) \quad \ln P (T_{j,i}, T_{j,i} + \delta_{j,i})]'$.

Finally, comparing, for $s = 1$, the right-hand-side of equation (4.36) with, for instance, Johnson and Kotz (1972, equation 35.4), it follows that, conditional on \mathcal{F}_t and under the forward measure $\mathcal{Q}^{T_{j+1}}$, the joint probability law of $\ln P (T_j, T_j + \delta_j)$ and $\ln P (T_{j,i}, T_{j,i} + \delta_{j,i})$ can be represented by the following bivariate normal density function:

$$(4.37) \quad \begin{aligned} & \mathcal{Q}^{T_{j+1}} [\ln P (T_j, T_j + \delta_j) \in dx \wedge \ln P (T_{j,i}, T_{j,i} + \delta_{j,i}) \in dy \mid \mathcal{F}_t] \\ &= \left[2\pi \sqrt{g (t, T_j, T_{j+1}) g (t, T_{j,i}, T_{j,i} + \delta_{j,i}) [1 - \rho^2 (T_j, T_{j,i})]} \right]^{-1} \exp \left\{ -\frac{1}{2 [1 - \rho^2 (T_j, T_{j,i})]} \right. \\ & \quad \left[\left(\frac{x - \mu (t, T_j, T_{j+1})}{\sqrt{g (t, T_j, T_{j+1})}} \right)^2 - 2\rho (T_j, T_{j,i}) \left(\frac{x - \mu (t, T_j, T_{j+1})}{\sqrt{g (t, T_j, T_{j+1})}} \right) \left(\frac{y - \mu (t, T_{j,i}, T_{j,i} + \delta_{j,i})}{\sqrt{g (t, T_{j,i}, T_{j,i} + \delta_{j,i})}} \right) \right. \\ & \quad \left. \left. + \left(\frac{y - \mu (t, T_{j,i}, T_{j,i} + \delta_{j,i})}{\sqrt{g (t, T_{j,i}, T_{j,i} + \delta_{j,i})}} \right)^2 \right] \right\} dx dy. \end{aligned}$$

Simplifying the above density with the help of identities (4.18) to (4.24), each expectation on the right-hand-side of equation (4.30) can be written as:

$$(4.38) \quad \begin{aligned} & E_{\mathcal{Q}^{T_{j+1}}} \left[\exp (-\ln P (T_j, T_j + \delta_j)) 1_{\{\ln(1+\delta_{j,i}r_u(T_{j,i}))^{-1} \leq \ln P (T_{j,i}, T_{j,i} + \delta_{j,i}) \leq \ln(1+\delta_{j,i}r_l(T_{j,i}))^{-1}\}} \mid \mathcal{F}_t \right] \\ &= \int_{\mathfrak{R}} dx \exp (-x) \int_{\ln(1+\delta_{j,i}r_u(T_{j,i}))^{-1}}^{\ln(1+\delta_{j,i}r_l(T_{j,i}))^{-1}} dy (2\pi \sqrt{\varepsilon_{j,i}})^{-1} \\ & \quad \exp \left[-\frac{1}{2} (a_{j,i}x^2 + b_{j,i}y^2 + c_{j,i}xy + d_{j,i}x + e_{j,i}y + f_{j,i}) \right]. \end{aligned}$$

Isolating all the terms in y , inside the last exponential, and completing the square,

$$(4.39) \quad \begin{aligned} & E_{\mathcal{Q}^{T_{j+1}}} \left[\exp (-\ln P (T_j, T_j + \delta_j)) 1_{\{\ln(1+\delta_{j,i}r_u(T_{j,i}))^{-1} \leq \ln P (T_{j,i}, T_{j,i} + \delta_{j,i}) \leq \ln(1+\delta_{j,i}r_l(T_{j,i}))^{-1}\}} \mid \mathcal{F}_t \right] \\ &= (2\pi \varepsilon_{j,i} b_{j,i})^{-\frac{1}{2}} \int_{\mathfrak{R}} dx \exp \left[-\frac{a_{j,i}x^2}{2} - \left(1 + \frac{d_{j,i}}{2} \right) x - \frac{f_{j,i}}{2} + \frac{(c_{j,i}x + e_{j,i})^2}{8b_{j,i}} \right] \\ & \quad \int_{\ln(1+\delta_{j,i}r_u(T_{j,i}))^{-1}}^{\ln(1+\delta_{j,i}r_l(T_{j,i}))^{-1}} dy \left(\frac{2\pi}{b_{j,i}} \right)^{-\frac{1}{2}} \exp \left[-\frac{1}{2} \frac{\left(y + \frac{c_{j,i}x + e_{j,i}}{2b_{j,i}} \right)^2}{\frac{1}{b_{j,i}}} \right]. \end{aligned}$$

Since the last integrand defines the density function of a univariate normal random variable with mean equal to $\left(-\frac{c_{j,i}x+e_{j,i}}{2b_{j,i}}\right)$ and variance $\frac{1}{b_{j,i}}$, and completing the square inside the first exponential,

$$\begin{aligned}
(4.40) \quad & E_{\mathcal{Q}^{T_{j+1}}} \left[\exp(-\ln P(T_j, T_j + \delta_j)) 1_{\{\ln(1+\delta_{j,i}r_u(T_{j,i}))^{-1} \leq \ln P(T_{j,i}, T_{j,i}+\delta_{j,i}) \leq \ln(1+\delta_{j,i}r_l(T_{j,i}))^{-1}\}} \middle| \mathcal{F}_t \right] \\
&= (2\pi\varepsilon_{j,i}b_{j,i})^{-\frac{1}{2}} \exp \left[\frac{e_{j,i}^2}{8b_{j,i}} - \frac{f_{j,i}}{2} + \frac{(c_{j,i}e_{j,i} - 4b_{j,i} - 2b_{j,i}d_{j,i})^2}{8b_{j,i}(4a_{j,i}b_{j,i} - c_{j,i}^2)} \right] \\
&\quad \int_{\mathbb{R}} dx \exp \left\{ -\frac{1}{2} \left[\left(\frac{4a_{j,i}b_{j,i} - c_{j,i}^2}{4b_{j,i}} \right)^{\frac{1}{2}} x - \left(\frac{4a_{j,i}b_{j,i} - c_{j,i}^2}{4b_{j,i}} \right)^{-\frac{1}{2}} \left(\frac{c_{j,i}e_{j,i} - 4b_{j,i} - 2b_{j,i}d_{j,i}}{4b_{j,i}} \right) \right]^2 \right\} \\
&\quad \left\{ \Phi \left[\frac{\ln(1 + \delta_{j,i}r_l(T_{j,i}))^{-1} + \frac{c_{j,i}x+e_{j,i}}{2b_{j,i}}}{\frac{1}{\sqrt{b_{j,i}}}} \right] - \Phi \left[\frac{\ln(1 + \delta_{j,i}r_u(T_{j,i}))^{-1} + \frac{c_{j,i}x+e_{j,i}}{2b_{j,i}}}{\frac{1}{\sqrt{b_{j,i}}}} \right] \right\}.
\end{aligned}$$

Performing an obvious change of variables and rearranging terms,

$$\begin{aligned}
(4.41) \quad & E_{\mathcal{Q}^{T_{j+1}}} \left[\exp(-\ln P(T_j, T_j + \delta_j)) 1_{\{\ln(1+\delta_{j,i}r_u(T_{j,i}))^{-1} \leq \ln P(T_{j,i}, T_{j,i}+\delta_{j,i}) \leq \ln(1+\delta_{j,i}r_l(T_{j,i}))^{-1}\}} \middle| \mathcal{F}_t \right] \\
&= (\varepsilon_{j,i}b_{j,i})^{-\frac{1}{2}} \exp \left[\frac{e_{j,i}^2}{8b_{j,i}} - \frac{f_{j,i}}{2} + \frac{(c_{j,i}e_{j,i} - 4b_{j,i} - 2b_{j,i}d_{j,i})^2}{8b_{j,i}(4a_{j,i}b_{j,i} - c_{j,i}^2)} \right] \left(\frac{4a_{j,i}b_{j,i} - c_{j,i}^2}{4b_{j,i}} \right)^{-\frac{1}{2}} \int_{\mathbb{R}} dz (2\pi)^{-\frac{1}{2}} \\
&\quad \exp \left(-\frac{1}{2}z^2 \right) \left\{ \Phi \left[\frac{\sqrt{b_{j,i}} \ln(1 + \delta_{j,i}r_l(T_{j,i}))^{-1} + \frac{\sqrt{b_{j,i}}(2a_{j,i}e_{j,i} - 2c_{j,i} - d_{j,i}c_{j,i})}{4a_{j,i}b_{j,i} - c_{j,i}^2} - \frac{\zeta z}{\sqrt{1-\zeta^2}}}{\sqrt{b_{j,i}} \ln(1 + \delta_{j,i}r_l(T_{j,i}))^{-1} + \frac{\sqrt{b_{j,i}}(2a_{j,i}e_{j,i} - 2c_{j,i} - d_{j,i}c_{j,i})}{4a_{j,i}b_{j,i} - c_{j,i}^2} - \frac{\zeta z}{\sqrt{1-\zeta^2}}} \right] \right. \\
&\quad \left. - \Phi \left[\frac{\sqrt{b_{j,i}} \ln(1 + \delta_{j,i}r_u(T_{j,i}))^{-1} + \frac{\sqrt{b_{j,i}}(2a_{j,i}e_{j,i} - 2c_{j,i} - d_{j,i}c_{j,i})}{4a_{j,i}b_{j,i} - c_{j,i}^2} - \frac{\zeta z}{\sqrt{1-\zeta^2}}}{\sqrt{b_{j,i}} \ln(1 + \delta_{j,i}r_u(T_{j,i}))^{-1} + \frac{\sqrt{b_{j,i}}(2a_{j,i}e_{j,i} - 2c_{j,i} - d_{j,i}c_{j,i})}{4a_{j,i}b_{j,i} - c_{j,i}^2} - \frac{\zeta z}{\sqrt{1-\zeta^2}}} \right] \right\},
\end{aligned}$$

where

$$(4.42) \quad \zeta := -\frac{c_{j,i}}{\sqrt{4a_{j,i}b_{j,i}}}.$$

Finally, the last integral can be computed explicitly by applying lemma 4.2, and therefore it is possible to convert equation (4.30) into the analytical pricing solution (4.16). ■

5. CONCLUSIONS

The main purpose and contribution of this paper consisted in deriving an exact analytical pricing formulae, under a multi-factor Gaussian HJM framework, for *floating range notes*. It is remarkable that the value of such interest rate correlation dependent assets only involves the univariate normal distribution function, no matter the dimension of the term structure model under consideration.

To the author's knowledge, proposition 4.3 constitutes the first generalization of the single-factor pricing solutions, previously obtained by Turnbull (1995) and Navatte and Quittard-Pinon (1999), towards a more realistic multi-factor setup. In addition, proposition 4.1 also offers an exact and explicit pricing solution for the simpler class of *fixed range notes*.

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