Multi-Factor and Analytical Valuation of Treasury Bond Futures with an Embedded Quality Option*

João Pedro Vidal Nunes[†]
CEMAF/ISCTE

Luís Alberto Ferreira de Oliveira
CEMAF/ISCTE

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[†]Corresponding author. Address: CEMAF/ISCTE, Complexo INDEG/ISCTE, Av. Prof. Aníbal Bettencourt, 1600-189 Lisboa, Portugal. Tel: +351 21 7958607. Fax: +351 21 7958605. E-mail: joao.nunes@iscte.pt

Abstract

A closed-form pricing solution is proposed for the quality option embedded in Treasury bond futures contracts, under a multi-factor and Gaussian Heath, Jarrow, and Morton (1992) framework. Such an analytical solution can be obtained through a *conditioning* approximation, in the sense of Curran (1994) and Rogers and Shi (1995), or via a *rank 1* approximation, following Brace and Musiela (1994). Monte Carlo simulations show that both approximations are extremely accurate and easy to calculate.

Application of the proposed pricing model to the EUREX market, from January 2000 through May 2004, yields an excellent fit and an insignificant estimate of the quality option magnitude. On average, this delivery option accounts for only 0.04% of the futures prices.

Key words: Gaussian HJM multi-factor models, Quality option, Consistent forward rate curves, Treasury bond futures, EUREX market.

JEL Classification: C15, E43, G13.

1 Introduction

Treasury bond futures contracts include a variety of features known as *delivery options* that give the party with a short position some flexibility concerning delivery. The two types of delivery options considered in the literature are the *quality option* and the *timing option*.[1]

The quality option allows the short position to deliver, on the delivery day, any of the deliverable bonds specified by the exchange. For each deliverable bond (of each delivery month) the exchange defines, *a priori*, a conversion factor that will adjust the invoice amount to be paid by the futures buyer in order to make the long position indifferent as to the choice of the deliverable issue. However, such conversion factors are computed as the clean prices of the deliverable bonds on the delivery day, so that the yields-to-maturity of all deliverable issues are equal to the notional coupon rate. Therefore, the conversion factors system is only able to define an approximate equivalence relation among the different deliverable bonds and, consequently, the futures buyer must be compensated for the additional delivery risk through a "bid-down" adjustment of the market futures price.

The measurement and modeling of the negative impact of delivery options on the market price of futures contracts has been the subject of extensive study by various methods, on different markets, and with different empirical results. The initial valuation approach assumed a deterministic interest rate setting where marking to market was ignored, so futures contracts were essentially valued as if they were forward contracts.[2] More recent models for the quality option take into account the stochastic nature of the term structure of interest rates, either though an equilibrium framework or via an arbitrage-free model.

Carr and Chen (1997) obtain pricing solutions for the quality option under a Cox, Ingersoll, and Ross (1985) framework, and apply their results to the Chicago Board of Trade (CBOT) Treasury bond (T-bond) futures contracts for 1987-1991, concluding that the quality option value can be large. Under a similar framework and for the same time period, Chen (1997) derives upper bounds for the delivery options embedded in Treasury bond futures contracts. He shows that delivery options can significantly affect futures prices, and the main impact is due to the quality option. Bick (1997) also derives an analytical pricing formula, under a Vasicek (1977) framework, but for futures on zero-coupon Treasury bonds.

Ritchken and Sankarasubramanian (1992) consider a one-factor Heath et al. (1992) (HJM hereafter) term structure model with a Vasicek (1977)-type volatility structure. They provide an exact closed-form solution for futures prices with an embedded quality option but on pure discount bonds, and price (numerically) futures contracts on coupon-bearing bonds. Ritchken and Sankarasubramanian (1995) later extended their single-factor model into a two-factor Gaussian HJM framework, where the quality option is valued numerically. Using CBOT T-bond futures contracts, they show that the quality option value can be significant (and higher than the one implied by a single-factor model): on average and three months prior to the delivery day, it is found to equal 0.1%, 1%, and 2% of contract size for two-, ten-, and fifteen-year futures contracts, respectively.

Lin and Paxson (1995) implement a one-factor Gaussian HJM model through a discrete-time binomial grid to price the German Bund futures contracts traded on the London International Financial Futures and Options Exchange (LIFFE) from December 1988 through November 1991. Their results show that the conventional quality option value implied in Bund futures contracts is much lower than the one usually found for the CBOT market, averaging only 9 basis points of contract size three months prior to delivery. Similarly, but under an extended Vasicek (1977) model, which is implemented through a discrete trinomial tree approach, Lin, Chen, and Chou (1999) show that the average quality option value associated with the Japanese long-term government bond futures contracts is only equal to about 0.02% of par three months before delivery. However, such insignificant estimates of the quality option value might arise not only because of specific contractual features, but because all bond returns are perfectly correlated under their single-factor frameworks. For instance, Santa-Clara and Sornette (2001, Table 4) find that the quality option implicit in the CBOT March 1999 long bond futures contract can be about four-times more valuable under a *string model* than under a model driven by a single Brownian motion.[3] Therefore, it makes sense to test the relevance of the quality option feature under more general correlation structures.

The contribution of this article to the interest rate futures literature is twofold. Firstly, this article proposes, for the first time to the authors' knowledge, an analytical valuation formula for the quality option embedded in Treasury bond futures contracts, under a multi-factor Gaussian HJM term structure model. This closed-form pricing solution establishes the value of a Treasury bond futures contract (with an embedded quality option) as a weighted average of zero-coupon bond futures prices, where the weights are expressed in terms of the univariate normal distribution function. Hence, the present research can be understood as an extension of Ritchken and Sankarasubramanian (1992).

Secondly, the proposed analytical pricing formula is also used to test the relevance of the quality option feature in the EUREX market from January 2000 through May 2004. As in Lin and Paxson (1995), but under a more robust multi-factor framework, it is shown that the quality option value has little impact on the German Treasury bond futures contracts.

2 Valuation framework

Hereafter, uncertainty will be represented by a filtered probability space $(\Omega, \mathcal{F}, \mathcal{Q}, \mathbb{F})$ satisfying the usual technical conditions, where $\mathcal{Q}$ is the risk-neutral measure obtained when the money-market account is taken to be the numéraire of the underlying continuous-time economy. Furthermore:

Assumption 1 *Futures contracts are assumed to be continuously (instead of daily) marked to market.*

Assumption 2 *There are no timing options.*[4]

2.1 Quality option concept

Under the previous assumptions and following, for instance, Ritchken and Sankarasubramanian (1992, equation 4.2) while borrowing the notation from Bühler, Düllmann, and Windfuhr (2001), it is well known that the time-t fair price of a bond futures contract with an embedded quality option, that matures at time T_f $(\geq t)$ and is written on a delivery basket containing m deliverable Treasury coupon-bearing bonds is equal to:

$$H\left(t, T_f, \{1, \ldots, m\}\right) = \mathbb{E}_{\mathcal{Q}} \left\{ \min_{j=1}^{m} \left[\frac{CB_j\left(T_f\right)}{cf_j} \right] \middle| \mathcal{F}_t \right\}, \tag{1}$$

where $CB_j\left(t\right)$ represents the time-t clean price of the j^{th} deliverable bond and cf_j is the corresponding conversion factor.

Ignoring the quality option value is equivalent to assuming that the choice of the bond issue to be delivered at time T_f is made irreversibly on the valuation date (time t). Therefore, again following Ritchken and Sankarasubramanian (1992, equation 4.3), the time-t fair price of a bond futures contract that matures at time T_f $(\geq t)$ and is written on a delivery basket containing m deliverable Treasury coupon-bearing bonds but that possesses no quality option, would be equal to

$$H\left(t, T_f, \{j^*\}\right) = \min_{j=1}^{m} \left\{ \mathbb{E}_{\mathcal{Q}} \left[\frac{CB_j\left(T_f\right)}{cf_j} \middle| \mathcal{F}_t \right] \right\}, \tag{2}$$

where $j^* = \arg\min_{j=1}^{m} \left\{ \mathbb{E}_{\mathcal{Q}} \left[\frac{CB_j(T_f)}{cf_j} \middle| \mathcal{F}_t \right] \right\}$ denotes the time-t cheapest-to-deliver bond.

The quality option value can now be obtained as the difference between the price of a futures contract allowing only one bond to be delivered (the current cheapest-to-deliver) and the price of a similar futures contract that allows several bonds to be delivered. This will be represented by

$$QO\left(t, T_f, \{1, \ldots, m\}\right) = H\left(t, T_f, \{j^*\}\right) - H\left(t, T_f, \{1, \ldots, m\}\right). \tag{3}$$

2.2 Multi-factor Gaussian HJM model

To derive an analytical solution for equation (3), a multi-factor HJM Gaussian term structure model will be adopted. The choice of the HJM framework is necessary in order to obtain a perfect fit to the market prices of all deliverable Treasury bonds. The use of a multi-factor specification is intended to enhance the model's fit to the Treasury bond futures market and to accommodate the usual prescription of three stylized factors in the principal components analysis of the yield curve: *level*, *slope*, and *curvature* (see, for instance, Litterman and Scheinkman (1991)).

The Gaussian HJM model under use can be formulated in terms of risk-free pure discount bond prices, which are assumed to evolve through time (under measure $\mathcal{Q}$) according to the following stochastic differential equation:

$$\frac{dP\left(t,T\right)}{P\left(t,T\right)} = r\left(t\right)dt + \underline{\sigma}\left(t,T\right)' \cdot d\underline{W}^{\mathcal{Q}}\left(t\right), \tag{4}$$

where $P\left(t,T\right)$ represents the time-t price of a unit face value and default-free zero-coupon bond expiring at time T $(\geq t)$; $r\left(t\right)$ is the time-t instantaneous spot rate; $\cdot$ denotes the inner product in $\Re^n$; and $\underline{W}^{\mathcal{Q}}\left(t\right) \in \Re^n$ is a standard Brownian motion, initialized at zero and generating the augmented, right-continuous, and complete filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq t_0\}$, where t_0 denotes the current time.

The n-dimensional adapted volatility function, $\underline{\sigma}\left(\cdot, T\right) : [t_0, T] \rightarrow \Re^n$, is assumed to satisfy the usual mild measurability and integrability requirements –as stated, for instance, in Lamberton and Lapeyre (1996, theorem 3.5.5)– as well as the "pull-to-par" condition $\underline{\sigma}\left(u,u\right) = \underline{0} \in \Re^n, \forall u \in [t_0, T]$. Moreover,

Assumption 3 *The volatility function $\underline{\sigma}\left(\cdot,\cdot\right)$ is assumed to be deterministic.*

This assumption is adopted for reasons of analytical tractability; that is, in order to obtain lognormally distributed pure discount bond prices, after applying Itô's lemma to equation (4):

$$\begin{aligned} P(t,T) &= P(t_0,t,T)\exp\left\{-\frac{1}{2}\int_{t_0}^{t}\left[\left\|\underline{\sigma}(s,T)\right\|^2-\left\|\underline{\sigma}(s,t)\right\|^2\right]ds \right. \\ &\quad \left. +\int_{t_0}^{t}\left[\underline{\sigma}(s,T)-\underline{\sigma}(s,t)\right]'\cdot d\underline{W}^{Q}(s)\right\}, \end{aligned} \tag{5}$$

where

$$P(t_0,t,T) := \frac{P(t_0,T)}{P(t_0,t)} \tag{6}$$

defines the time-t_0 forward price for delivery at time t of a risk-free pure discount bond with maturity at time T.

Nevertheless, the proposed multi-factor Gaussian HJM model is not necessarily Markovian or time-homogeneous. Moreover, Pang (1998, Subsection 1.1.2) shows that the proposed framework (and therefore all the forthcoming analytical pricing solutions) can be easily generalized in a Gaussian random field term structure model, in the sense of Kennedy (1994).

As shown, for example, by El Karoui, Lepage, Myneni, Roseau, and Viswanathan (1991, equation 47), a closed-form solution is easily obtained using the HJM model (4) under consideration for a Treasury bond futures contract without any delivery option. Specifically, equation (2) can be rewritten as

$$H(t_0,T_f,\{j^*\}) = \min_{j=1}^{m}\left[-\frac{AI_j(T_f)}{cf_j}+\sum_{i=1}^{N^j}F\left(t_0,T_f,T_i^j\right)\frac{k_i^j}{cf_j}\right], \tag{7}$$

where N^j is the number of cash flows k_i^j $(i=1,\ldots,N^j)$ paid by the j^{th} underlying coupon-bearing bond at times T_i^j $(>T_f)$; cf_j is the conversion factor of such coupon-bearing bond; $AI_j(T_f)$ denotes its time-T_f accrued interest; and

$$F(t_0,T_f,T) = P(t_0,T_f,T)\exp\left[-\frac{1}{2}\eta(t_0,T_f,T)+\frac{1}{2}\varphi(t_0,T_f,T)\right] \tag{8}$$

with

$$\eta(t_0,T_f,T) := \int_{t_0}^{T_f}\left[\left\|\underline{\sigma}(s,T)\right\|^2-\left\|\underline{\sigma}(s,T_f)\right\|^2\right]ds \tag{9}$$

and

$$\varphi(t_0,T_f,T) := \int_{t_0}^{T_f}\left\|\underline{\sigma}(s,T)-\underline{\sigma}(s,T_f)\right\|^2 ds \tag{10}$$

represents the time-t_0 price of a futures contract for delivery at time T_f $(\geq t_0)$ and on a default-free zero-coupon bond with expiration at time T $(\geq T_f)$, with $\|\cdot\|$ denoting the Euclidean norm in $\Re^n$.

The main purpose of this article is to generalize the pricing solution (7) to accommodate the existence of a quality option.

3 Treasury bond futures with an embedded quality option

In order to compute the expectation contained in equation (1), it is necessary to consider the transition probability density function, under the $\mathcal{Q}$ martingale measure, for all the stochastic factors underlying the terminal (time-T_f) deliverable coupon-bearing bond prices. However, the number of such stochastic variables increases with the dimension of vector $\underline{W}^{\mathcal{Q}}(t)$ and with the number of cash flow payment dates generated by the deliverable basket under consideration.

The first difficulty –the number of Brownian motions under consideration– could be overcome by using a simpler one-dimensional model, but the cost of this simplification would certainly be a much poorer fit to the observable futures prices. The second problem –dependence on the cash flow structure of the delivery basket– could also be solved by using a simpler factor-model rather than an arbitrage-free term structure framework. However, adoption of general equilibrium term structure specifications does not guarantee the model fit to the market spot prices of all the deliverable bonds.

As an alternative, the arbitrage-free and multi-factor model defined in equation (4) is adopted, and two different approximations are proposed in order to reduce the dimensionality of the integration problem implicit in equation (1). The first is based on the *conditioning approach* initiated by Curran (1994), Rogers and Shi (1995), and Nielsen and Sandmann (2003) in the context of Asian option pricing, and extended by Nielsen and Sandmann (2002) to a stochastic interest rate setting.[5] The second is based on the *proportionality* (or *rank 1*) *assumption* that El Karoui and Rochet (1989) and Brace and Musiela (1994) use to price European options on coupon-bearing bonds. In both cases, assumption 3 is required in order to ensure analytical tractability, no matter the number of Brownian motions or the dimension of the delivery basket under consideration.

3.1 Conditioning approach

The next lines of discussion develop an analytical solution for the price of a futures contract with an embedded quality option in the context of the conditioning approach.

3.1.1 Upper bound

Following Rogers and Shi (1995), let $Z \in \Re$ be any $\mathcal{F}_{T_f}$-measurable random variable. From the law of iterative expectations and using Jensen's inequality, equation (1) can be rewritten as

$$\begin{aligned} H\left(t_0, T_f, \{1, \ldots, m\}\right) &= \mathbb{E}_{\mathcal{Q}}\left\{\mathbb{E}_{\mathcal{Q}}\left[\min_{j=1}^{m}\left(\frac{CB_j\left(T_f\right)}{cf_j}\right)\middle| Z\right]\middle| \mathcal{F}_{t_0}\right\} \\ &\leq H^u\left(t_0, T_f, \{1, \ldots, m\}\right), \end{aligned} \tag{11}$$

where

$$H^u\left(t_0, T_f, \{1, \ldots, m\}\right) := \mathbb{E}_{\mathcal{Q}}\left\{\min_{j=1}^{m}\left[\mathbb{E}_{\mathcal{Q}}\left(\frac{CB_j\left(T_f\right)}{cf_j}\middle| Z\right)\right]\middle| \mathcal{F}_{t_0}\right\} \tag{12}$$

defines an upper bound for the true futures price. The next proposition provides an explicit solution for the conditional expectation on the right-hand-side of equation (12) by assuming a standard normal distribution for the conditioning variable.

Proposition 1 *Under the HJM model (4), the time-t_0 fair price, $H\left(t_0, T_f, \{1, \ldots, m\}\right)$, of a bond futures contract that matures at time T_f $(\geq t_0)$ and is written on a delivery basket containing m deliverable Treasury coupon-bearing bonds is bounded from above by*

$$H^u\left(t_0, T_f, \{1, \ldots, m\}\right) = \mathbb{E}_{\mathcal{Q}}\left\{\min_{j=1}^{m}\left[\frac{CB_j\left(T_f; Z\right)}{cf_j}\right]\middle| \mathcal{F}_{t_0}\right\}, \tag{13}$$

with

$$\frac{CB_j\left(T_f; z\right)}{cf_j} := -\frac{AI_j\left(T_f\right)}{cf_j} + \sum_{i=1}^{N^j} \frac{k_i^j P\left(t_0, T_f, T_i^j\right)}{cf_j} \exp\left[-\frac{\tilde{\eta}\left(t_0, T_f, T_i^j\right)}{2} + \tilde{\varphi}\left(t_0, T_f, T_i^j\right) z\right], \tag{14}$$

where N^j is the number of cash flows k_i^j $(i = 1, \ldots, N^j)$ paid by the j^{th} deliverable bond at times T_i^j $(> T_f)$; cf_j is the conversion factor of such coupon-bearing bond; $AI_j\left(T_f\right)$ denotes its time-T_f accrued interest; $Z \backsim N^1(0,1)$;[6]

$$\tilde{\varphi}\left(t_0, T_f, T_i^j\right) := \mathbb{E}_{\mathcal{Q}}\left\{Z \int_{t_0}^{T_f}\left[\underline{\sigma}\left(s, T_i^j\right) - \underline{\sigma}\left(s, T_f\right)\right]' \cdot d\underline{W}^{\mathcal{Q}}(s)\middle| \mathcal{F}_{t_0}\right\}; \tag{15}$$

and

$$\tilde{\eta}\left(t_0, T_f, T_i^j\right) := \eta\left(t_0, T_f, T_i^j\right) - \varphi\left(t_0, T_f, T_i^j\right) + \tilde{\varphi}\left(t_0, T_f, T_i^j\right)^2. \tag{16}$$

Proof. See Appendix A. ■

Notice that whatever the number of Brownian motions or the number of cash flow payment dates generated by the deliverable basket under consideration, proposition 1 reduces the valuation of the futures contract to a univariate integration problem. In order to obtain an analytical solution for equation (13), the following notation will be useful.

Definition 1 *Denote by z_k^*, $k = 1, \ldots, r$, all possible solutions in z for the following set of non-linear equations:*

$$\frac{CB_j (T_f; z)}{cf_j} = \frac{CB_l (T_f; z)}{cf_l}, \quad j = 1, \ldots, m-1, \quad l = j+1, \ldots, m.$$

Furthermore, set $z_0^* = -\infty$ and $z_{r+1}^* = \infty$. All roots are assumed to be arranged in increasing order, i.e., $z_1^* < z_2^* < \ldots < z_r^*$.

The next theorem describes the main contribution of this research, namely, an approximate analytical pricing solution for Treasury bond futures contracts with embedded quality options, in the context of a multi-factor HJM Gaussian model.

Theorem 1 *Under the assumptions of proposition 1:*

$$\begin{aligned} & H^u (t_0, T_f, \{1, \ldots, m\}) \\ = & \sum_{j=1}^{m} \frac{AI_j (T_f)}{cf_j} \sum_{k=1}^{r+1} I_{j,k} \left[\Phi \left(z_{k-1}^* \right) - \Phi \left(z_k^* \right) \right] + \sum_{j=1}^{m} \sum_{i=1}^{N^j} \frac{k_i^j}{cf_j} F \left(t_0, T_f, T_i^j \right) w_{j,i}, \end{aligned} \tag{17}$$

where $\Phi(\cdot)$ represents the cumulative density function of the univariate standard normal distribution; z_k^*, $k = 0, \ldots, r+1$, are given by definition 1; $I_{j,k}$ is the indicator function

$$I_{j,k} := \mathbf{1}_{\left\{ \frac{CB_j \left(T_f; z \right)}{cf_j} \leq \frac{CB_l \left(T_f; z \right)}{cf_l}, \forall l \neq j, \forall z \in \left[z_{k-1}^*, z_k^* \right] \right\}}; \tag{18}$$

and

$$w_{j,i} := \sum_{k=1}^{r+1} I_{j,k} \left\{ \Phi \left[z_k^* - \tilde{\varphi} \left(t_0, T_f, T_i^j \right) \right] - \Phi \left[z_{k-1}^* - \tilde{\varphi} \left(t_0, T_f, T_i^j \right) \right] \right\}. \tag{19}$$

Proof. See Appendix B. ■

As in Carr and Chen (1997, equation 15), $w_{j,i}$ can be understood as weights, such that $\sum_{j=1}^{m} w_{j,i} = 1, \forall i$. Therefore, the analytical pricing solution (17) can also be interpreted as essentially a weighted average of pure discount bond futures prices.[7]

3.1.2 Conditioning variable

The pricing formula (17) will become a completely explicit solution only after specification of the conditioning random variable Z, which will define the deterministic function $\tilde{\varphi}\left(t_0, T_f, T_i^j\right)$. As Nielsen and Sandmann (2002, page 360) argue, it is not possible to find the conditioning variable Z that minimizes the approximation error $H^u\left(t_0, T_f, \{1, \ldots, m\}\right) - H\left(t_0, T_f, \{1, \ldots, m\}\right)$, i.e., with a perfect correlation with $\min_{j=1}^{m}\left[\frac{CB_j(T_f)}{cf_j}\right]$. Hence, Z will be chosen simply in order to enhance such correlation. Nevertheless, notice that even under the worst scenario of zero correlation, the upper bound $H^u\left(t_0, T_f, \{1, \ldots, m\}\right)$ would be identical to the futures price without the quality option, as given by equation (7).

The next proposition defines Z as being perfectly correlated (up to a first-order approximation) with $\frac{1}{m}\sum_{j=1}^{m}\frac{CB_j(T_f)}{cf_j}$. Therefore, the approximate pricing solution in theorem 1 can be understood as arising from conditioning the minimum of all deliverable bonds' time-T_f clean prices, corrected by their conversion factors, on the corresponding arithmetic average, in the same way as Curran (1994) conditions the arithmetic average on the corresponding geometric mean.[8]

Proposition 2 *Under the assumptions of proposition 1, if*

$$Z = \frac{1}{\alpha}\sum_{u=1}^{m}\sum_{v=1}^{N^u}\frac{k_v^u}{cf_u}P\left(t_0, T_f, T_v^u\right)\int_{t_0}^{T_f}\left[\underline{\sigma}\left(s, T_v^u\right) - \underline{\sigma}\left(s, T_f\right)\right]' \cdot d\underline{W}^{\mathcal{Q}}(s), \tag{20}$$

where

$$\alpha^2 := \sum_{u=1}^{m}\sum_{v=1}^{N^u}\sum_{p=1}^{m}\sum_{q=1}^{N^p}\frac{k_v^u k_q^p}{cf_u cf_p}P\left(t_0, T_f, T_v^u\right)P\left(t_0, T_f, T_q^p\right)\psi\left(t_0, T_f, T_v^u, T_q^p\right) \tag{21}$$

with

$$\psi\left(t_0, T_f, T_v^u, T_q^p\right) := \int_{t_0}^{T_f}\left[\underline{\sigma}\left(s, T_v^u\right) - \underline{\sigma}\left(s, T_f\right)\right]' \cdot \left[\underline{\sigma}\left(s, T_q^p\right) - \underline{\sigma}\left(s, T_f\right)\right] ds, \tag{22}$$

then

$$Z \backsim N^1(0, 1), \tag{23}$$

$$\tilde{\varphi}\left(t_0, T_f, T_i^j\right) = \frac{1}{\alpha}\sum_{u=1}^{m}\sum_{v=1}^{N^u}\frac{k_v^u}{cf_u}P\left(t_0, T_f, T_v^u\right)\psi\left(t_0, T_f, T_v^u, T_i^j\right), \tag{24}$$

and[9]

$$corr\left[\frac{1}{m}\sum_{j=1}^{m}\frac{CB_j(T_f)}{cf_j}, Z\right] \approx 1. \tag{25}$$

Proof. Condition (23) arises from definition (20) after noticing that $\alpha^2 = \mathbb{E}_{\mathcal{Q}}\left[(\alpha Z)^2\middle|\mathcal{F}_{t_0}\right]$. Equation (24) follows immediately from definitions (15) and (20). Finally, property (25) is based on the first-order approximation $\exp(x) \approx 1 + x$. Details available upon request. ■

3.2 Proportionality assumption

The second approximation proposed to value futures with an embedded quality option is based on the following result:

Proposition 3 *Under the HJM model (4), the time-t price of a pure discount bond with maturity T $(\geq t)$ is equal in distribution, under the equivalent martingale measure $\mathcal{Q}$, to:*

$$P(t,T) \stackrel{d}{=} P(t_0,t,T) \exp\left[-\frac{1}{2}\eta(t_0,t,T) + \sqrt{\varphi(t_0,t,T)}z\right], \tag{26}$$

with $z \backsim N^1(0,1)$.

Proof. Equation (26) arises after applying Arnold (1992, corollary 4.5.6) to equation (5). ■

An approximate analytical pricing solution can be obtained for Treasury bond futures by assuming that equation (26) is valid not only as an equality in distribution, but also as an equality in value. Such an approximation is consistent with the proportionality assumption of El Karoui and Rochet (1989, page 22) and with the rank 1 approximation suggested by Brace and Musiela (1994, equation 6.1), both shown to be accurate in the context of European swaption pricing; see, for example, Brace and Musiela (1994, Table 7.5). Although it is based on completely different assumptions, the next corollary shows that such an approximate pricing formula has the same structure as the one obtained in theorem 1 under the conditioning approach.

Corollary 1 *Assuming that equation (26) is valid not only as an equality in distribution but also as an equality in value, the time-t_0 fair price, $H(t_0, T_f, \{1,\ldots,m\})$, of a bond futures contract that matures at time T_f $(\geq t_0)$ and is written on a delivery basket containing m deliverable Treasury coupon-bearing bonds can be approximated through theorem 1, replacing $\tilde{\varphi}\left(t_0, T_f, T_i^j\right)$ by $\sqrt{\varphi\left(t_0, T_f, T_i^j\right)}$, for $j = 1,\ldots,m$ and $i = 1,\ldots,N^j$.*

Proof. Approximating equation (26) as an equality in value, equation (1) can be rewritten as:

$$\begin{aligned} & H\left(t_0, T_f, \{1, \ldots, m\}\right) \\ \approx \; & \mathbb{E}_{\mathcal{Q}}\left\{ \min_{j=1}^{m} \left[\sum_{i=1}^{N^j} \frac{k_i^j P\left(t_0, T_f, T_i^j\right) \exp\left(-\frac{\eta\left(t_0, T_f, T_i^j\right)}{2} + \sqrt{\varphi\left(t_0, T_f, T_i^j\right)} z\right)}{cf_j} - \frac{AI_j\left(T_f\right)}{cf_j} \right] \middle| \mathcal{F}_{t_0} \right\}. \end{aligned} \tag{27}$$

Comparing equations (13) and (27), corollary 1 follows. ■

From a theoretical point of view, there are at least two reasons for the proportionality assumption to be less appealing than the conditioning approach: First, the approximation error of the former methodology should increase with the number of factors. Second, the sign of the approximation error is unknown. Corollary 1 implies that the proposed approximations would be equivalent if, and only if,

$$\tilde{\varphi}\left(t_0, T_f, T_i^j\right) = \sqrt{\varphi\left(t_0, T_f, T_i^j\right)}, \forall i, j. \tag{28}$$

Combining equations (21) and (24):

$$\begin{aligned} & \tilde{\varphi}\left(t_0, T_f, T_i^j\right)^2 \\ = \; & \frac{\sum_{u=1}^{m} \sum_{v=1}^{N^u} \sum_{p=1}^{m} \sum_{q=1}^{N^p} \frac{k_v^u k_q^p P\left(t_0, T_f, T_v^u\right) P\left(t_0, T_f, T_q^p\right)}{cf_u cf_p} \psi\left(t_0, T_f, T_v^u, T_i^j\right) \psi\left(t_0, T_f, T_q^p, T_i^j\right)}{\sum_{u=1}^{m} \sum_{v=1}^{N^u} \sum_{p=1}^{m} \sum_{q=1}^{N^p} \frac{k_v^u k_q^p P\left(t_0, T_f, T_v^u\right) P\left(t_0, T_f, T_q^p\right)}{cf_u cf_p} \psi\left(t_0, T_f, T_v^u, T_q^p\right)}, \end{aligned}$$

and noticing that $\psi\left(t_0, T_f, T_i^j, T_i^j\right) = \varphi\left(t_0, T_f, T_i^j\right)$, it follows that, if

$$\psi\left(t_0, T_f, T_v^u, T_i^j\right) \psi\left(t_0, T_f, T_q^p, T_i^j\right) \approx \psi\left(t_0, T_f, T_v^u, T_q^p\right) \varphi\left(t_0, T_f, T_i^j\right), \forall i, j, u, v, p, q, \tag{29}$$

the equivalence condition (28) will be obtained. Equation (29) is exact only under a single-factor specification ($n = 1$), but a Monte Carlo study will show that both approximations are almost indistinguishable under a three-factor model, for different parameter constellations and for different contract specifications.

3.3 Nested time-homogeneous specification

Although theorem 1 and corollary 1 are valid for any deterministic volatility specification (even for time-inhomogeneous or non-Markovian models), the subsequent empirical analysis will be cast in a simpler time-independent framework. Since the goal is only to fit market prices of Treasury bond futures, such a time-homogeneous setup should be sufficient to recover the main principal diagonal elements of the market interest rate covariance matrix.[10]

Following, for instance, Musiela and Rutkowski (1998, proposition 13.3.2), it is well known that if the short-term interest rate is Markovian and the volatility function $\underline{\sigma}(\cdot, T) : [t_0, T] \to \Re^n$ is time-homogeneous, then the volatility function must be restricted to the analytical specification:

$$\underline{\sigma}(t, T)' := \underline{G}' \cdot a^{-1} \cdot \left[I_n - e^{a(T-t)} \right], \tag{30}$$

where $I_n \in \Re^{n \times n}$ represents an identity matrix, while $\underline{G} \in \Re^n$ and $a \in \Re^{n \times n}$ express the model's time-independent parameters.

The Gauss-Markov time-homogeneous HJM model to be estimated is now defined by equations (4) and (30).[11]

3.4 Monte Carlo study

A Monte Carlo experiment is run to test the accuracy of the approximate analytical pricing solutions proposed in theorem 1 and corollary 1. For that purpose, forward zero-coupon bond prices are subject to a Euler discretization (under measure $\mathcal{Q}$) and evolved, from the valuation date (time t_0) and until the expiration date of the futures contract (time T_f), for all the maturities T ($\geq T_f$) that correspond to cash flow payment dates of all the underlying deliverable bonds. On each time step, a set of n independent normal and antithetic variates is generated using the Box-Muller algorithm. For all futures contracts tested below, Monte Carlo simulations are run with 520 time steps per year and until achieving an accuracy rate (ratio between the standard error and the price estimate) of one basis point.

The inputs needed to run the Monte Carlo study (initial yield curve, volatility parameters, and deliverable bonds) are taken from the dates within our data set for which the quality option value estimates are found to be higher: December 20, 2000 for a medium-term futures contract (with Bloomberg code OEU1 and two deliverable bonds); December 27, 2000 for the long-term contract RXM1 (with three deliverable bonds); and, May 11, 2001 for the short-term contract DUU1 (with seven deliverable bonds). Because the quality option value is low for all the selected dates, and in order to test the accuracy of the proposed approximations under different market conditions, a phantom futures contract (labeled ADUU1) was also created through the aggregation of the delivery baskets underlying the OEU1, RXM1 and DUU1 contracts (and with the same delivery date as the short-term contract). This artificial contract is valued not only on May 11, 2001, but also one year earlier (May 11, 2000) in order to further enhance the corresponding quality option value.[12]

Table I presents the results of the Monte Carlo study for a three-dimensional model specification. For each date, futures prices (with embedded quality options) are approximated through theorem 1 and proposition 2 (upper bound price) as well as via corollary 1 (rank 1 price). For all the contracts, both approximations are extremely fast to implement and the difference between their estimates (column rank 1 diff.) is completely negligible.[13]

In order to gauge the importance of the quality option feature on each date, each futures price is also computed assuming no delivery options, i.e. via equation (7). The quality option is then expressed as the percentage difference between the later exact futures price (without quality option features) and the futures price computed through the conditioning approach (upper bound column).

Pricing errors are defined as the differences between approximate (upper bound column) and exact (Monte Carlo) futures prices with embedded quality options. For all the dates considered, the pricing errors obtained are very small (under two basis points of the exact price) and similar in magnitude to the Monte Carlo standard errors. This impressive accuracy is observed not only for the traded contracts (OEU1, RXM1, and DUU1), where the quality option value is low (no higher than 1.11%), but also for the artificial contract ADUU1. Hence, the accuracy of the proposed approximations is not confined to the EUREX futures contracts, which are characterized by a narrow delivery basket and a short time-to-maturity. On May 11, 2001, the ADUU1 futures contract (with 15 deliverable issues) presents a much higher quality option value: 12.23%. The same contract valued on May 11, 2000, but based on model parameters and yield curve data prevailing on May 11, 2001, that is, with one additional year of time-to-maturity, yields an even higher estimate for the quality option: 25.62%. Nevertheless, for all dates and for all contracts, the two proposed approximations are shown to be remarkably accurate and efficient.

The precision of the approximation derived from the conditioning approach (which will be used in the subsequent empirical analysis) is justified by the judicious choice of the conditioning variable: the correlation coefficients between the conditioning variable and the terminal futures price are found to be very close to one for all the contracts under analysis.

Insert Table I about here.

4 Consistent forward rate curves

Before proceeding to the analysis of the EUREX market, this section applies the methodology proposed by Björk and Christensen (1999) to the term structure model under consideration.

4.1 Restrictions on the volatility function

To price interest rate contingent claims under the term structure model (4), two inputs are required: the initial forward rate curve and the parameters' values defining the volatility function (30).

Concerning the first input, many different parametric functional forms can be used to estimate forward rates from the observed market prices of coupon-bearing Treasury bonds; see, for instance, Jeffrey, Linton, and Nguyen (2000) for a survey. However, Björk and Christensen (1999) argue that the family $\mathcal{G}$ of forward rate curves used for model recalibration must be consistent with the dynamics implied by the interest rate model $\mathcal{M}$ under consideration.

Björk and Christensen (1999, page 327) mention two reasons for this consistency requirement. First, if a given interest rate model $\mathcal{M}$ is subject to daily recalibration, it is important that, on each day, the parameterized family of forward rate curves $\mathcal{G}$, which is fitted to bond market data, is general enough to be invariant under the dynamics of the term structure model; otherwise, the marking to market of an interest rate derivative would yield value changes attributable not to interest rate movements, but rather to model inconsistencies. Second, if a specific family $\mathcal{G}$ of forward rate curves can efficiently represent the cross-section of bond prices observed in the market, it makes sense to incorporate this implied yield behavior in the dynamics of the interest rate model $\mathcal{M}$ used. In practical terms, and as proposition 4 reveals, consistency between $\mathcal{M}$ and $\mathcal{G}$ can be ensured by incorporating in the volatility function (30) some parameters (matrix a) that are estimated through the best fitting of the initially observed forward rate curve.

Proposition 4 *Under the assumption that matrix a is diagonal,*[14] *the minimal consistent family $\mathcal{G}$ of forward rate curves that is invariant under the dynamics of the Gaussian and time-homogeneous HJM model specified by equations (4) and (30) is defined through the mapping* $\gamma : \Re^{2n} \times \Re_+ \rightarrow \Re$, *such that*

$$\gamma\left(\underline{z},x\right):=f\left(t,t+x\right)=\sum_{j=1}^{n}z_j\exp\left(a_jx\right)+\sum_{j=1}^{n}z_{n+j}\exp\left(2a_jx\right), \tag{31}$$

where z_j represents the j^{th} element of vector $\underline{z}\in\Re^{2n}$; $f\left(t,t+x\right)$ corresponds to the time-t instantaneous forward interest rate for date $(t+x)$, with $x\in\Re_+$; and a_j defines the j^{th} principal diagonal element of matrix a.

Proof. Equation (31) arises as a straightforward application of the two locally invariance conditions imposed by Björk and Christensen (1999, theorem 4.1). Details available upon request. ■

Equipped with the linear-exponential specification of the forward rate curve provided by proposition 4, parameters a and $\underline{z}$ can be estimated by minimizing the sum of absolute percentage differences between a cross-section of Treasury coupon-bearing bond prices and the discounted values obtained by decomposing each government bond into a portfolio of pure discount bonds.

4.2 Bond data set: Description and empirical results

The data used to estimate the spot yield curve were collected from Bloomberg. They consist of bid and ask prices, recorded at the end of each exchange session, for all the 144 German Treasury coupon-bearing bonds traded on the EUREX and on the Frankfurt Stock Exchange between January 3, 2000 and May 31, 2004.

To mitigate problems associated with distorted prices from bonds not actively traded during the sample period, the robust outlier identification procedure proposed in Rousseeuw (1990) is used to exclude from the sample all issues with a too large bid-ask spread.[15]

On each sample day, the term structure of interest rates is estimated by finding the values of the parameters $\underline{z}\in\Re^{2n}$ and $\{a_1,\ldots,a_n\}$ that minimize the mean absolute percentage errors (MAPE) or differences between fitted and market coupon-bearing bond prices. Throughout the empirical analysis, all optimization routines are based on the quasi-Newton method, with backtracking line searches, described in Dennis and Schnabel (1996, section 6.3). Following the usual principal components analysis prescription, the HJM model dimension is set at three factors ($n=3$). Hence, nine parameters are used in the forward rate curve specification (31).

The estimation methodology adopted is very good at fitting the discount function implicit in the German government bond market, resulting in reliable and smooth yield curves for the entire sample period: the sample average MAPE equals 0.06%, which is of the same magnitude as the average in-sample bid-ask spread. Figure 1 shows the estimated spot yield surface for maturities between one and fifteen years, with only two observations per month (on Wednesdays) during the sample period. Throughout almost the entire sample period, the spot yield curve presented a positive slope, although it was approximately flat between August and November 2000. Nevertheless, it is clear that the period under analysis includes a wide variety of term structure shapes.

Insert Figure 1 about here.

5 Empirical analysis of the quality option: The EUREX market

This section estimates the implicit quality option value embedded in the EUREX Treasury bond futures contracts during the sample period between January 3, 2000 and May 31, 2004, using the HJM framework presented in the previous sections.

5.1 Futures data set description

Futures prices and deliverable baskets were generously provided by the EUREX Statistical Department for short-term (Schatz), medium-term (Bobl), and long-term (Bund) contracts. For all these contracts, the underlying instrument corresponds to a notional German Treasury bond with a 6 percent (annual) coupon and a term of 1.75-2.25 years, 4.5-5.5 years, or 8.5-10.5 years, respectively.[16] Over $1,119$ trading days, a total of $9,327$ average bid-ask prices were gathered for 52 different futures contracts.

There are two main reasons for focusing the empirical analysis on EUREX Treasury bond futures contracts: their extreme liquidity, and the fact that they include no other embedded delivery options beyond the quality option. The EUREX Euro-Bund future is one of the world's most heavily traded futures contracts. Accordingly to EUREX statistics, in 2001, the Euro-Bund future accounted for more than 178 million contracts traded (see EUREX (2002)). During the same period, the CBOT traded approximately 58 million US T-bond futures contracts, and Euronext traded only 7 million Euro notional futures contracts. Moreover, the Schatz and the Bobl futures are the world's most heavily traded contracts in the 2- and 5-year segments.

5.2 Estimation methodology

For each sample day, and using the already-estimated term structure of interest rates, the HJM volatility parameters $\underline{G}$ are inferred by minimizing the mean absolute percentage differences between model futures values, as given by theorem 1 and proposition 2, and market prices. Figure 2 shows that the time-homogeneous volatility function (30) produces a remarkable fit between model futures values (with embedded quality options) and observed market prices: the average daily absolute percentage pricing error is only about 18 basis points (ranging from 0.005% to 0.626%). Then, the estimated spot yield curve and the volatility function (implicit in the market for Treasury bond futures contracts) are used to compute futures values without (that is, not deducted from) quality option features through equation (7). On each day, and for each futures contract, the quality option value is given by the percentage difference between equations (17) and (7).

Insert Figure 2 about here.

Before applying the previously described estimation methodology, some data-mining problems were dealt with: namely, liquidity deficiencies of the futures contracts (measured through traded volumes) and the treatment of the so-called *new issue quality option*.

In general, liquidity is observed to be practically nil from the issue date of the futures contract to thirty weeks prior to the delivery day. After this time, liquidity increases substantially and rapidly until contract maturity. Whenever there are no trades for a futures contract, the representativeness of its price must be questioned because prices are then computed by EUREX using the cost-of-carry model.[17] As noticed by El Karoui et al. (1991, page 13), and assuming a positive and increasing volatility specification, the forward price overestimates the true Treasury bond futures value, so it should not be used to price the quality option. Moreover, Chen (1997, proposition 5) also argues that futures prices are bounded from above by the cost-of-carry model. To obviate this problem, daily volumes of all futures contracts were examined and any prices that were not produced by market trades (1,124 observations) were excluded from the analysis. Table II presents the descriptive statistics for the quality option estimates, both with zero-volume prices excluded and also with the entire sample considered. Although the average value of the quality option is very low in both series, it is clear that the series including the zero-volume prices underestimates the value of the quality option.

Insert Table II about here.

The second problem concerns the existence of what Lin and Paxson (1995) call the new issue quality option, which arises whenever the exchange allows new bond issues to be included in the futures contracts delivery basket after the first trading day. This is the case for the Treasury bond futures contracts traded on EUREX, so the final list of deliverable bonds (and corresponding conversion factors) can be known only on the last trading day. Consequently, on each day the quality option value is derived not only from the possibility of choice among the currently known delivery set but, to less of an extent, also from the probability that new bond issues will be included in the delivery basket. Because it is not possible to know in advance which new bond issues might be added to the deliverable set, each futures contract is valued, on each trading day, based only on the currently known deliverable set.

5.3 Empirical results

The results in Table II show that, compared to other empirical studies of the US Treasury bond futures market, the EUREX quality option is quite irrelevant. Its global average value is only about 0.037% of the futures price (ranging from zero to a maximum of 1.11%), well within the daily average futures pricing absolute percentage error (see Figure 2). Moreover, our estimates yield a highly significant number of zero values for the quality option (the median is equal to -4.08×10^{-12}), which reinforces the assertion that the quality option contractual feature has an insignificant impact on the market futures prices.

Table II also decomposes the quality option estimates by contract type. The long-term contract has the highest quality option value: 1.11%. The short-term contract presents the lowest estimated average quality option value –only about 1.99 basis points– which is consistent with the findings of Ritchken and Sankarasubramanian (1995).

One possible explanation for the insignificance of the quality option estimates is the presence of a small number of homogeneous deliverable issues underlying the EUREX Treasury bond futures contracts. Between a minimum of two and a maximum of twelve, our sample includes an average of only four coupon-bearing deliverable bonds per contract. For instance, Hemler (1990) describes more than thirty US Treasury deliverable bonds (varying widely in terms of coupon rates and time-to-maturity) for the CBOT T-bond futures contracts. To validate the sparseness of deliverable bonds as an explanation for the small average quality option value found in the EUREX market, the quality option value of each futures contract was regressed against the number of deliverable bonds underlying the contract. For all the contract types and sub-samples tested, the slope of the regression was always found to be positive and statistically significant (almost always at the 1% level).[18]

As another plausible explanation, it can also be argued that the EUREX quality option is a European-style option (because there is only one pre-specified delivery day for each contract), while the CBOT quality option is a Bermudan option (since the futures seller can set the delivery day as being any trading day of the delivery month) and, therefore, more valuable.

6 Conclusions

The main theoretical contribution of this article is to derive a closed-form (but approximate) pricing solution for the quality option embedded in Treasury bond futures contracts, under a multi-factor Gaussian HJM framework. Using a conditioning or a rank 1 approximation, and no matter the diversity of the underlying delivery basket or the dimension of the HJM model under analysis, the fair value of a Treasury bond futures contract (with an embedded quality option) was written as a weighted average of pure discount bond futures prices, where the weights simply involve the standard univariate normal distribution function.

An extensive Monte Carlo experiment was run to test the accuracy of the approximate analytical pricing solutions proposed. For different parameter constellations and for different contract specifications, the pricing errors obtained were very low (and similar in magnitude to the Monte Carlo standard errors) as well as almost indistinguishable in either approximation.

Finally, the significance of the quality option was tested in the EUREX market through calibration of a three-factor and time-homogeneous HJM specification to the Treasury bond futures contracts traded between January 3, 2000 and May 31, 2004. Two notable empirical findings are the excellent fit to the market prices of Treasury bond futures contracts and the low average values estimated for the quality option. The empirical evidence suggests that the size of the quality option is quite irrelevant for the EUREX market, which contradicts the conclusions in most empirical studies of the US T-bond futures contracts. Nevertheless, the proposed pricing methodology should prove useful for other Treasury bond futures markets, as well as for the valuation of multi-issue bond futures contracts. Given space constraints, both topics are left for further research.

A Appendix: Proof of proposition 1

Decomposing a coupon-bearing bond into a portfolio of pure discount bonds, then:

$$\mathbb{E}_{\mathcal{Q}}\left[\left.\frac{CB_j\left(T_f\right)}{cf_j}\right| Z\right] = -\frac{AI_j\left(T_f\right)}{cf_j} + \sum_{i=1}^{N^j} \frac{k_i^j}{cf_j}\mathbb{E}_{\mathcal{Q}}\left[\left.P\left(T_f, T_i^j\right)\right| Z\right]. \tag{32}$$

To compute the expectation contained on the right-hand-side of equation (32), equations (5) and (9) can be combined into

$$\begin{aligned}\mathbb{E}_{\mathcal{Q}}\left[\left.P\left(T_f, T_i^j\right)\right| Z\right] &= P\left(t_0, T_f, T_i^j\right)\exp\left[-\frac{1}{2}\eta\left(t_0, T_f, T_i^j\right)\right] \\ &\quad \mathbb{E}_{\mathcal{Q}}\left\{\left.\exp\left[\int_{t_0}^{T_f}\left(\underline{\sigma}\left(s, T_i^j\right) - \underline{\sigma}\left(s, T_f\right)\right)' \cdot d\underline{W}^{\mathcal{Q}}(s)\right]\right| Z\right\}. \end{aligned} \tag{33}$$

Assuming that Z follows a standard univariate normal distribution, since

$$\int_{t_0}^{T_f}\left[\underline{\sigma}\left(s, T_i^j\right) - \underline{\sigma}\left(s, T_f\right)\right]' \cdot d\underline{W}^{\mathcal{Q}}(s) \backsim N^1\left(0, \varphi\left(t_0, T_f, T_i^j\right)\right)$$

and following, for example, Bartoszyński and Niewiadomska-Bugaj (1996, theorem 9.10.4), then:

$$\begin{aligned}&\left.\int_{t_0}^{T_f}\left[\underline{\sigma}\left(s, T_i^j\right) - \underline{\sigma}\left(s, T_f\right)\right]' \cdot d\underline{W}^{\mathcal{Q}}(s)\right| Z \\ \backsim\ & N^1\left(\tilde{\varphi}\left(t_0, T_f, T_i^j\right)Z, \varphi\left(t_0, T_f, T_i^j\right) - \tilde{\varphi}\left(t_0, T_f, T_i^j\right)^2\right),\end{aligned} \tag{34}$$

where the deterministic function $\tilde{\varphi}\left(t_0, T_f, T_i^j\right)$ is defined by the covariance (15).

Applying result (34) and using the definition of the moment-generating function of a normal random variable, equation (33) becomes

$$\begin{aligned}\mathbb{E}_{\mathcal{Q}}\left[\left.P\left(T_f, T_i^j\right)\right| Z\right] &= P\left(t_0, T_f, T_i^j\right)\exp\left[-\frac{1}{2}\eta\left(t_0, T_f, T_i^j\right)\right] \\ &\quad \exp\left\{\tilde{\varphi}\left(t_0, T_f, T_i^j\right)Z + \frac{1}{2}\left[\varphi\left(t_0, T_f, T_i^j\right) - \tilde{\varphi}\left(t_0, T_f, T_i^j\right)^2\right]\right\}.\end{aligned} \tag{35}$$

Combining equations (12), (32), and (35), the quasi-analytical solution (13) follows for the upper bound of the futures price.[19] ■

B Appendix: Proof of theorem 1

Following Ritchken and Sankarasubramanian (1992, page 212), equation (13) can be rewritten as:

$$\begin{aligned} H^u\left(t_0, T_f, \{1, \ldots, m\}\right) &= \sum_{j=1}^{m} \int_{z \in S_j} dz \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right) \left\{ -\frac{AI_j\left(T_f\right)}{cf_j} \right. \\ &\left. + \sum_{i=1}^{N^j} \frac{k_i^j P\left(t_0, T_f, T_i^j\right) \exp\left[-\frac{\tilde{\eta}\left(t_0, T_f, T_i^j\right)}{2} + \tilde{\varphi}\left(t_0, T_f, T_i^j\right) z\right]}{cf_j} \right\}, \end{aligned}$$

where

$$S_j := \left\{ z \in \Re : \frac{CB_j\left(T_f; z\right)}{cf_j} \leq \frac{CB_l\left(T_f; z\right)}{cf_l}, \forall l \neq j \right\} \tag{36}$$

can be interpreted as the region of the state space where the j^{th} deliverable bond is the time-T_f cheapest-to-deliver. Since the roots z_k^*, $k = 0, \ldots, r+1$, are assumed to be arranged in increasing order, and using equation (18), then

$$\begin{aligned} H^u\left(t_0, T_f, \{1, \ldots, m\}\right) &= \sum_{j=1}^{m} \sum_{k=1}^{r+1} I_{j,k} \int_{z_{k-1}^*}^{z_k^*} dz \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right) \left\{ -\frac{AI_j\left(T_f\right)}{cf_j} \right. \\ &\left. + \sum_{i=1}^{N^j} \frac{k_i^j P\left(t_0, T_f, T_i^j\right) \exp\left[-\frac{\tilde{\eta}\left(t_0, T_f, T_i^j\right)}{2} + \tilde{\varphi}\left(t_0, T_f, T_i^j\right) z\right]}{cf_j} \right\}. \end{aligned}$$

Expressing all integrands in terms of the univariate normal density function and using equation (8), equation (17) follows. ■

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Notes

[1]The timing option is usually found to be less important than the quality option and arises whenever the futures seller is allowed to select the moment (during the delivery month) that delivery takes place. This is the case for the CBOT futures contracts but not for the EUREX bond futures covered by this paper, since for the latter, delivery must take place on the tenth calendar day of each (quarterly) delivery month.

[2]Such an approach includes, for instance, the analysis of Gay and Manaster (1984), Barnhill and Seale (1988), Hegde (1988), Barnhill (1990), and Hemler (1990), all devoted to the CBOT futures contracts.

[3]Santa-Clara and Sornette (2001) ignore the timing option feature by assuming a delivery date at the middle of the delivery month.

[4]This assumption is consistent with the contractual features of the Treasury bond futures traded on EUREX, which will be used in the forthcoming empirical analysis.

[5]The authors thank Klaus Sandmann for suggesting this approach.

[6]Hereafter, the notation $X \backsim N^1\left(\mu, \sigma^2\right)$ is intended to mean that the one-dimensional random variable X is normally distributed, with mean μ and variance σ^2.

[7]Carr and Chen (1997) propose a similar representation involving the non-central chi-square distribution function, but under a one-factor Cox et al. (1985) model.

[8]The conversion factors' scheme should ensure a positive correlation between $\min_{j=1}^{m}\left[\frac{CB_j(T_f)}{cf_j}\right]$ and $\frac{1}{m}\sum_{j=1}^{m}\frac{CB_j(T_f)}{cf_j}$ by reducing the dispersion of the m underlying random variables.

[9]$corr\left(X, Y\right)$ represents the linear correlation coefficient between the random variables X and Y.

[10]Of course, as Rebonato (1998, page 70) argues, it will be extremely difficult to fit the market interest rate correlation structure through a low-dimensional and time-independent HJM Gauss-Markov model. Because our purpose is not to price or hedge interest rate correlation-dependent derivatives, the time-homogeneity restriction should not be too severe.

[11]Using the volatility specification (30), it is possible to obtain explicit solutions for several time integrals contained in the previous results. Formulae available upon request.

[12]The additional conversion factors needed are computed as the unit face value clean prices of the deliverable bonds, on the delivery date, such that the yields-to-maturity of all deliverable issues are equal to 6%. Details available upon request.

[13]All computations were made by running Pascal programs on an Intel Xeon 2.80 GHz processor.

[14]As argued by Duan and Simonato (1999, page 131), such "assumption of diagonability does not involve an appreciable loss of generality" because the eigenvalues of a matrix are continuous functions of its elements, and thus, multiple roots of the characteristic equation can be avoided by a small adjustment in the original matrix. Moreover, such restrictions on the off-principal diagonal elements of matrix a will prove useful to ensure the stability of the parameters' estimates.

[15]On each day, the bid-ask spreads of all traded bonds are standardized using the sample median (location estimator) and the median of all absolute deviations from the sample median (scale estimator). Whenever the standardized score of a specific bond is higher than a pre-specified cutoff value, defined as 2.5, the bond is automatically excluded from the cross-section under analysis. Furthermore, Treasury bonds with a time-to-maturity of less than three months are also excluded from the sample in order to alleviate illiquidity problems usually associated with issues that are close to their maturity date. After application of both liquidity filters, the average sample size of each cross-section decreased from 72 to 53 bonds.

[16]The less liquid Euro-Buxl contract is excluded from the forthcoming empirical analysis.

[17]This model treats futures as forward contracts, since it assumes that the futures price equals the spot price of the cheapest-to-deliver bond minus the unobtainable carry return (coupons), and plus the avoided carrying cost (i.e., the cost of funding the purchase of the underlying).

[18]Details available upon request.

[19]Quasi-analytical in the sense that it involves a single integration over the domain of Z.

Table I: Monte Carlo Study

Valuation date	Futures contracts	Analytical solutions				Antithetic Monte Carlo			
		Exact price without QO	Approx. price with QO: Upper bound	Approx. price with QO: Rank 1 diff.	Quality option	Exact price with QO	Standard errors	Pricing errors	Correlation $(H(T_f), Z)$
20 Dec 00	OEU1	105.16	104.02	0.0008	1.10%	104.02	0.010	-0.001	0.9850
			(0.01 s)	(0.02 s)		(10,337.25 s)			
27 Dec 00	RXM1	109.11	107.91	0.0010	1.11%	107.92	0.013	-0.014	0.9576
			(0.06 s)	(0.05 s)		(62,197.11 s)			
11 May 01	DUU1	102.58	101.94	0.0002	0.63%	101.92	0.010	0.016	0.9998
			(0.05 s)	(0.03 s)		(12.58 s)			
11 May 01	ADUU1	100.90	89.90	-0.0007	12.23%	89.88	0.009	0.017	0.9769
			(0.49 s)	(0.36 s)		(14,253.22 s)			
11 May 00	ADUU1	95.18	75.77	-0.0004	25.62%	75.76	0.008	0.012	0.9755
			(1.16 s)	(1.05 s)		(152,940.32 s)			

This table reports the results from the Monte Carlo study of the quality option (QO) value at four different valuation dates and for four different futures contracts (which are identified by their Bloomberg codes). Contract ADUU1 is fictitious and corresponds to the enlargement of the delivery basket associated with the traded contract DUU1. The last line values the artificial contract ADUU1 one year earlier, although based on the model parameters and the spot yield curve prevailing on May 11, 2001. Exact futures price without QO is given by equation (7). Approximate analytical futures prices with QO are given by theorem 1 for the upper bound, and by corollary 1 for the rank 1 approximation. The column titled "Rank 1 diff." presents the difference between the two approximated futures prices. The quality option is expressed as the percentage difference between the exact futures price without QO and the upper bound for the futures price with QO. Monte Carlo simulations are run with 520 time steps per year and until an accuracy (ratio between the standard error and the price estimate) of one basis point is achieved. Pricing errors are the differences between the upper bound analytical approximation and exact Monte Carlo futures prices. The CPU times are reported in parentheses and expressed in seconds. The last column shows the correlation coefficient between the simulated conditioning variable (Z) and the simulated futures value at the delivery date, i.e.: $H(T_f) \equiv \min_{j=1}^{m} \left[\frac{CB_j(T_f)}{cf_j} \right]$.

Table II: Quality Option Estimates

	Complete sample	Restricted sample			
		Global	Short-term	Medium-term	Long-term
Panel A: All futures' maturities considered					
Average	0.0249%	0.0367%	0.0199%	0.0346%	0.0525%
Median	-4.08E-12	-4.08E-12	-4.08E-12	-4.08E-12	-8.57E-14
Standard deviation	0.0600%	0.0985%	0.0642%	0.0873%	0.1254%
Minimum	-5.14E-12	-1.96E-11	-1.96E-11	-8.33E-12	-4.56E-12
Maximum	0.7592%	1.1081%	0.6302%	1.0988%	1.1081%
No. observations	9,327	8,203	2,462	2,748	2,993
Panel B: Average quality option estimates by time-to-maturity					
No. weeks to delivery:					
20		0.0535%	0.0312%	0.0498%	0.0703%
15		0.0413%	0.0254%	0.0460%	0.0521%
10		0.0349%	0.0188%	0.0329%	0.0529%
9		0.0172%	0.0053%	0.0130%	0.0335%
8		0.0157%	0.0054%	0.0115%	0.0303%
7		0.0210%	0.0104%	0.0137%	0.0388%
6		0.0216%	0.0135%	0.0157%	0.0357%
5		0.0144%	0.0085%	0.0104%	0.0243%
4		0.0101%	0.0056%	0.0058%	0.0189%
3		0.0067%	0.0040%	0.0050%	0.0110%
2		0.0034%	0.0003%	0.0009%	0.0089%
1		0.0024%	0.0000%	0.0007%	0.0066%

This table reports the quality option estimates obtained for the EUREX market from January 3, 2000 to May 31, 2004. The complete sample includes all the 52 Treasury bond futures contracts that were traded at EUREX over the sample period. The restricted sample is obtained by neglecting all zero-volume observations. The quality option value is expressed as the percentage difference between futures prices without delivery options (computed via equation (7)) and futures prices with an embedded quality option (as given by theorem 1 and proposition 2). The last three columns decompose the quality option estimates by contract type: short-term (Schatz), medium-term (Bobl) and long-term (Bund) futures contracts. Panel B presents the evolution of the average quality option value as a function of time-to-delivery, for the restricted sample and by contract type.

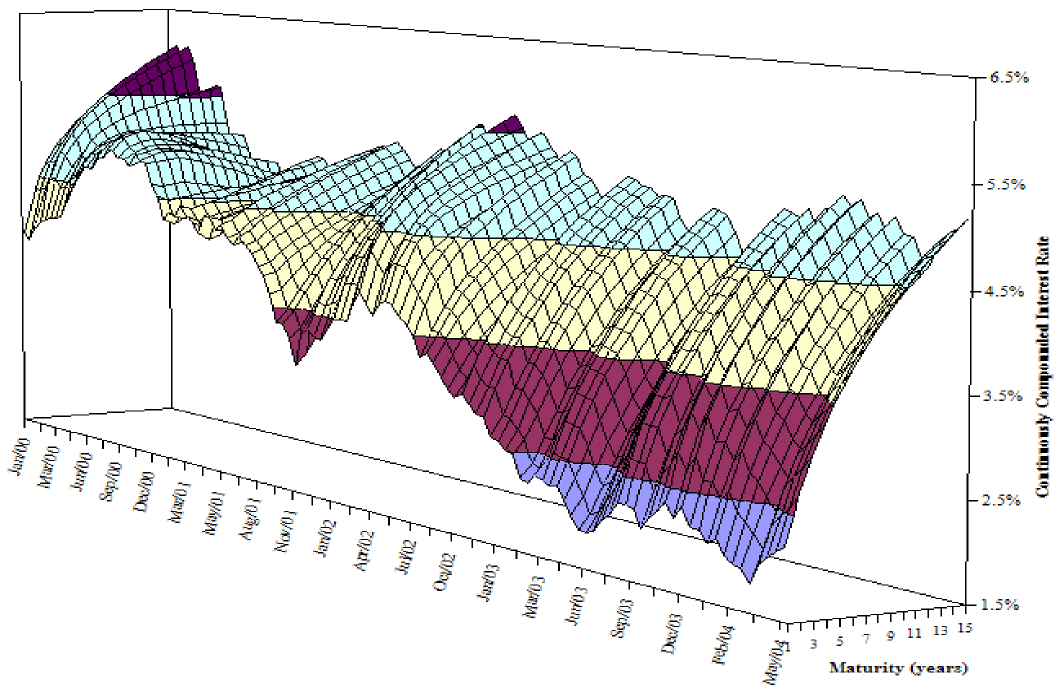

Figure 1: Spot yield surface

This figure shows the continuously compounded spot rates estimated for maturities between one and fifteen years from January 3, 2000 to May 31, 2004 for the German Treasury bond market, using the forward rate curve specification of equation (31). Only two observations per month (on Wednesdays) are presented.

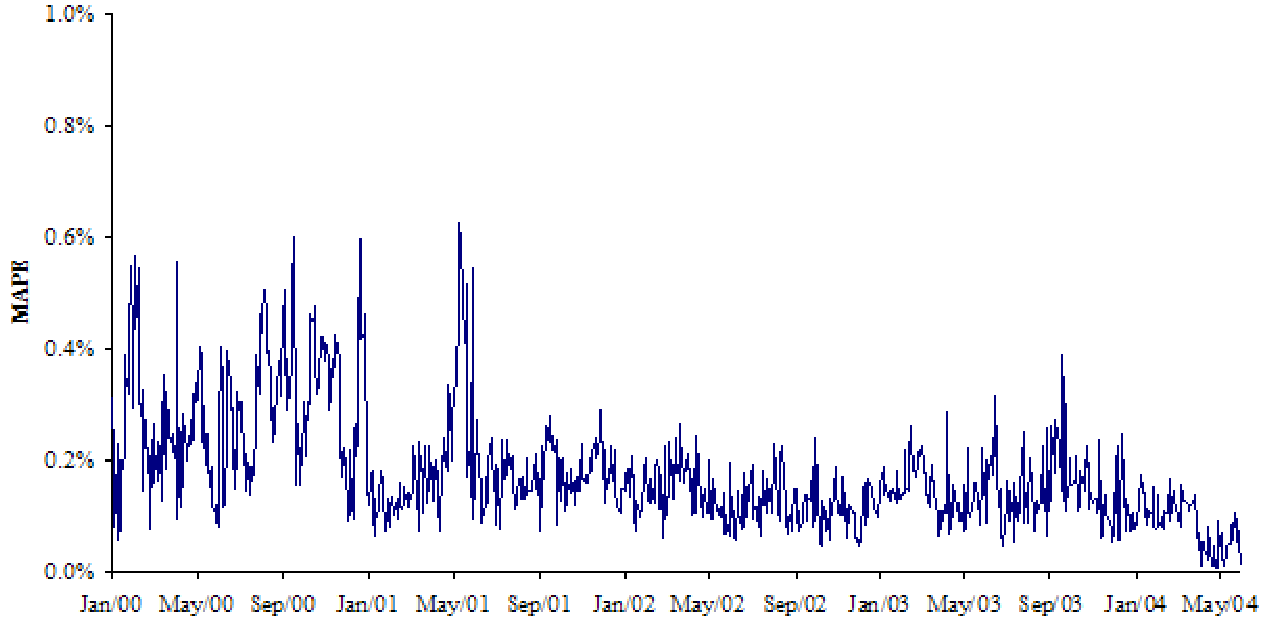

Figure 2: Model fit to the EUREX Treasury bond futures market

This figure shows in-sample mean absolute percentage errors for the EUREX Treasury bond futures market (zero-volume contracts excluded), from January 3, 2000 to May 31, 2004. On each sample day, the mean absolute percentage error (MAPE) corresponds to the average of the absolute percentage differences between each futures contracts' fair value (as given by theorem 1 and proposition 2) and market price.