COMPLEMENTOS DE OPÇÕES 2003-2004
MESTRADO EM FINANÇAS - ISCTE
EXAME - Resolução

13/07/04 **Duração: 2.5 horas**

<u>CASO 1</u>

a) Defina uma estratégia de *static hedging* para uma carteira com duas opções sobre idêntico activo subjacente, com igual *strike* e igual data de vencimento: uma posição *long European cash-or-nothing put* com um *contract size* igual ao *strike*; e uma posição *short European asset-or-nothing put* com um *contract size* igual a EUR1.

Valor da carteira na data de vencimento (momento "T"):

$$\begin{cases} X \Leftarrow S_T < X \\ 0 \Leftarrow S_T \geq X \end{cases} - EUR1 \times \begin{cases} S_T \Leftarrow S_T < X \\ 0 \Leftarrow S_T \geq X \end{cases} = \begin{cases} X - S_T \Leftarrow S_T < X \\ 0 \Leftarrow S_T \geq X \end{cases} = p_T(S_T; X; T)$$

Estratégia de *static hedging*: comprar uma *standard European put* sobre igual activo, com igual strike e igual data de vencimento.

b) A obrigação JPN tem vencimento a 1 ano, reembolso *bullet* e um cupão anual igual a 10% do valor nominal, caso a acção JPN desça abaixo de 80% da cotação actual em qualquer momento durante o próximo ano. Caso contrário, o cupão será igual a zero. Formule a avaliação desta obrigação.

O *payoff*, na data de vencimento, da obrigação ACN é dado por:

$$B_1 = 100\% + \begin{cases} 10\% \Leftarrow \inf_{u \in [0,1]} (S_u) < 0.8 S_0 \\ 0\% \Leftarrow else \end{cases}$$

$$\Leftrightarrow B_1 = 100\% + 10\% \times 1_{\left\{ \inf_{u \in [0,1]} (S_u) < 0.8 S_0 \right\}}$$

Portanto,

$$B_0 = 100\% \times e^{-r \times 1} + 10\% \times e^{-r \times 1} \times \Pr_0^Q \left\{ \inf_{u \in [0,1]} \left[\ln\left(\frac{S_u}{S_0} \right) \right] < \ln(0.8) \right\}$$

$$= 100\% \times e^{-r \times 1} + 10\% \times e^{-r \times 1} \times \left(1 - \Pr_0^Q \left\{ \inf_{u \in [0,1]} \left[\ln\left(\frac{S_u}{S_0} \right) \right] > \ln(0.8) \right\} \right)$$

Via equação (122),

$$B_0 = 110\% \times e^{-r\times 1} - 10\% \times e^{-r\times 1} \times \Pr_0^Q \left\{ \inf_{u\in[0,1]} \left[\ln\left(\frac{S_u}{S_0} \right) \right] > \ln(0.8) \right\},$$

sendo:

$$\Pr_0^Q \left\{ \inf_{u\in[0,1]} \left[\ln\left(\frac{S_u}{S_0} \right) \right] > \ln(0.8) \right\} = \Phi\left[d_2^M(1;0.8)\right] - (0.8)^{\frac{2\mu}{\sigma^2}} \times \Phi\left[d_2^M(0.8;1)\right].$$

c) Comente a seguinte afirmação: "O valor de uma *complex chooser option* pode ser decomposto numa carteira de *standard options* e de *compound put options*". Justifique.

Afirmação verdadeira.

Via equação (68):

$$AYLI_0 = c_t\left[c_t\left(S_t; X_c; T_{2c}\right); 0; T_1\right] + c_t\left[p_t\left(S_t; X_p; T_{2p}\right); 0; T_1\right].$$

Utilizando a paridade *put-call* para *compound options* (vide equação (41) e (42)):

$$\begin{aligned} AYLI_0 = {} & p_t\left[c_t\left(S_t; X_c; T_{2c}\right); 0; T_1\right] + c_t\left(S_t; X_c; T_{2c}\right) \\ & + p_t\left[p_t\left(S_t; X_p; T_{2p}\right); 0; T_1\right] + p_t\left(S_t; X_p; T_{2p}\right). \end{aligned}$$

CASO 2

a)

$$\mathrm{r}: \quad \mathrm{e}^{\mathrm{r}\times 6/12} = 1 + 2.25\% \times 6/12 \Rightarrow \mathrm{r} = \frac{12}{6} \ln\left(1 + 2.25\% \times 6/12\right) \cong 2.237\%.$$

$$\mathrm{S}_{5,9} = 2{,}275.17 \times \exp\left\{ \left[2.237\% + 2.31\% - \frac{(0.16)^2}{2} \right] \times \frac{1}{12} + (-1.2798) \times 0.16 \times \sqrt{\frac{1}{12}} \right\}$$

$$\cong 2{,}142.17.$$

$$\max\left(S_{i,8}\right) < 3{,}000 \Rightarrow \quad V_{6,8} = \max(2{,}700 - 2{,}679.69; 0) = 20.31.$$

$$\left(\mathrm{V}_{6,8}\right)^2 = (20.31)^2 \cong 412.33.$$

b)

$$\hat{V}_0 = e^{-2.237\%\times 0.5} \times \frac{550.48 + 730.88 + 228.01 + 20.31 + 623.87}{10}$$
$$= e^{-3.47\%\times 0.5} \times \frac{2{,}153.55}{10}$$
$$\cong 212.96.$$

303,028.42+534,188.48+51,989.30+412.33+389,215.13 = 1,278,833.65

$$\sigma\left(\hat{V}_0\right) = \frac{e^{-2.237\%\times 0.5}}{\sqrt{10}} \times \sqrt{\frac{1{,}278{,}833.65 - {(2{,}153.55)^2}/{10}}{10-1}} \cong 94.11.$$

c)

Valor actual do depósito bancário:

$$B_0 = \frac{100\%}{1 + 2.25\% \times {6}/{12}} + RV_0.$$

Por seu turno,

$$RV_{6M} = 60\% \times \begin{cases} 4\% \Leftarrow \dfrac{S_0 - S_{6M}}{S_0} \geq 4\% \wedge S_i < 3{,}000, \forall i \in \{1M, 2M, ..., 6M\} \\ \dfrac{S_0 - S_{6M}}{S_0} \Leftarrow 0\% \leq \dfrac{S_0 - S_{6M}}{S_0} \leq 4\% \wedge S_i < 3{,}000, \forall i \in \{1M, 2M, ..., 6M\} \\ 0\% \Leftarrow \dfrac{S_0 - S_{6M}}{S_0} \leq 0\% \vee \exists i \in \{1M, 2M, ..., 6M\} : S_i \geq 3{,}000 \end{cases}$$

$$\Leftrightarrow RV_{6M} = \frac{60\%}{S_0} \times \begin{cases} 0.04S_0 \Leftarrow S_{6M} \leq 0.96S_0 \wedge S_i < 3{,}000, \forall i \in \{1M, 2M, ..., 6M\} \\ S_0 - S_{6M} \Leftarrow S_0 \geq S_{6M} \geq 0.96S_0 \wedge S_i < 3{,}000, \forall i \in \{1M, 2M, ..., 6M\} \\ 0 \Leftarrow S_{6M} > S_0 \vee \exists i \in \{1M, 2M, ..., 6M\} : S_i \geq 3{,}000 \end{cases}$$

$$\Leftrightarrow RV_{6M} =$$

$$\frac{60\%}{S_0}\times\begin{cases} S_0 - S_{6M} \Leftarrow S_{6M} \leq S_0 \wedge S_i < 3{,}000, \forall i \in \{1M, 2M, ..., 6M\} \\ \\ 0 \Leftarrow S_{6M} > S_0 \vee \exists i \in \{1M, 2M, ..., 6M\} : S_i \geq 3{,}000 \end{cases}$$

$$-\frac{60\%}{S_0}\times\begin{cases} 0.96S_0 - S_{6M} \Leftarrow S_{6M} \leq 0.96S_0 \wedge S_i < 3{,}000, \forall i \in \{1M, 2M, ..., 6M\} \\ \\ 0 \Leftarrow S_{6M} > 0.96S_0 \vee \exists i \in \{1M, 2M, ..., 6M\} : S_i \geq 3{,}000 \end{cases}$$

$$\Leftrightarrow RV_{6M} = \frac{60\%}{S_0}\times\left[p_{6M}^{uo}(S_{6M}; X = S_0; H = 3{,}000; T = 6M) - p_{6M}^{uo}(S_{6M}; X = 0.96S_0; H = 3{,}000; T = 6M)\right]$$

Portanto,

$$RV_0 = \frac{60\%}{S_0}\times\left[p_0^{uo}(S_0; X = 2{,}700; H = 3{,}000; T = 6M) - p_0^{uo}(S_0; X = 2{,}592; H = 3{,}000; T = 6M)\right]$$

$$\Leftrightarrow RV_0 = \frac{60\%}{2{,}700}\times\left[212.96 - p_0^{uo}(S_0; X = 2{,}592; H = 3{,}000; T = 6M)\right]$$

Agora, basta avaliar a segunda *OTM European up-and-out put*:

European Up-And-Out OTM put

H	3,000.00
X	2,592.00

j	**max($S_{i,j}$)**	**$V_{T,j}$**
1	3,062.55	0.00
2	2,531.41	442.48
3	2,476.43	622.88
4	2,736.76	0.00
5	3,307.25	0.00
6	3,142.68	0.00
7	2,754.16	120.01
8	2,679.69	0.00
9	2,488.41	515.87
10	3,044.11	0.00
	Sum	1,701.24

$$\mathrm{RV}_0 = \frac{60\%}{2{,}700} \times \left[212.96 - \frac{1{,}701.24 / 10}{1 + 2.25\% \times 0.5} \right]$$

$$\mathrm{RV}_0 = \frac{60\%}{2{,}700} \times (212.96 - 168.23) \cong 0.994\%.$$

$$\mathrm{B}_0 = 98.89\% + 0.994\% = 99.88\% < 100\% \Rightarrow \text{ Não investir.}$$

CASO 3

a)

$$r: \quad e^r = 1 + 5.54846\% \Rightarrow r = \ln(1 + 5.54846\%) \cong 5.40\%.$$

$$B_0 = 100\% \times e^{-5.40\% \times 1} + RV_0.$$

$$\begin{aligned} RV_1 &= 2\% + \begin{cases} 10\% \Leftarrow \dfrac{S_1 - S_0}{S_0} < -20\% \vee \dfrac{S_1 - S_0}{S_0} > 20\% \\ 0\% \Leftarrow ELSE \end{cases} \\ &= \left(12\% - \begin{cases} 0 \Leftarrow ELSE \\ 10\% \Leftarrow 0.8S_0 \leq S_{0.5} \leq 1.2S_0 \end{cases} \right) \\ &= \left[12\% - 10\% \times RD_1(S; X_a = 0.8S_0; X_b = 1.2S_0; T = 1; M = 1) \right]. \end{aligned}$$

Mas,

$$RD_0(S; X_a = 0.8S_0; X_b = 1.2S_0; T = 1; M = 1)$$

$$= e^{-5.40\%} \times \left\{ N\left[d_2^M(0.8S_0)\right] - N\left[d_2^M(1.2S_0)\right] \right\}$$

$$= D(1)_0(S; X = 0.8S_0; T = 1; M = 1) - D(1)_0(S; X = 1.2S_0; T = 1; M = 1)$$

$$= D(1)_0(S; X = 3{,}552; T = 1; M = 1) - D(1)_0(S; X = 5{,}328; T = 1; M = 1)$$

$$= 0.82 - 0.17 = 0.65.$$

Em suma,

$$RV_0 = 12\% \times e^{-5.40\% \times 1} - 10\% \times 0.65 \cong 4.87\%.$$

$B_0 = 94.74\% + 4.87\% = 99.61\% < 100\% \Rightarrow$ Não depositar.

b)

Via proposição 22:

$$AYLI_0 = 4{,}440 \times e^{-3.76\% \times 1} \times M\left(a_1, b_{1c}; \sqrt{\frac{0.5}{1}}\right) - 4{,}440 \times e^{-5.40\% \times 1} \times M\left(a_2, b_{2c}; \sqrt{\frac{0.5}{1}}\right)$$
$$+ 4{,}000 \times e^{-5.40\% \times 1} \times M\left(-a_2, -b_{2p}; \sqrt{\frac{0.5}{1}}\right) - 4{,}440 \times e^{-3.76\% \times 1} \times M\left(-a_1, b_{1p}; \sqrt{\frac{0.5}{1}}\right)$$

Visto que:

$\overline{S} = 4{,}169.41;$

$$a_1 = \frac{\ln\left(\frac{4{,}440}{4{,}169.41}\right) + \left(5.40\% - 3.76\% + \frac{(0.2)^2}{2}\right) \times 0.5}{0.2 \times \sqrt{0.5}} \cong 0.573316;$$

$$a_2 = 0.573316 - 0.2 \times \sqrt{0.5} \cong 0.431895;$$

$$b_{1c} = \frac{\ln\left(\frac{4{,}440}{4{,}440}\right) + \left(5.40\% - 3.76\% + \frac{(0.2)^2}{2}\right) \times 1}{0.2 \times \sqrt{1}} \cong 0.182;$$

$$b_{2c} = 0.182 - 0.2 \times \sqrt{1} \cong -0.018;$$

$$b_{1p} = \frac{\ln\left(\frac{4{,}440}{4{,}000}\right) + \left(5.40\% - 3.76\% + \frac{(0.2)^2}{2}\right) \times 1}{0.2 \times \sqrt{1}} \cong 0.7038;$$

$$b_{2p} = 0.7038 - 0.2 \times \sqrt{1} \cong 0.5038;$$

então:

$$\begin{aligned} AYLI_0 &= 4{,}440 \times e^{-3.76\% \times 1} \times M(0.573316, 0.182; 0.707107) \\ &- 4{,}440 \times e^{-5.40\% \times 1} \times M(0.431895, -0.018; 0.707107) \\ &+ 4{,}000 \times e^{-5.40\% \times 1} \times M(-0.431895, -0.5038; 0.707107) \\ &- 4{,}440 \times e^{-3.76\% \times 1} \times M(-0.573316, 0.7038; 0.707107) \end{aligned}$$

Utilizando a tabela de probabilidades,

$$\begin{aligned} AYLI_0 &= 4{,}440 \times e^{-3.76\% \times 1} \times 0.515056 - 4{,}440 \times e^{-5.40\% \times 1} \times 0.43933 \\ &+ 4{,}000 \times e^{-5.40\% \times 1} \times 0.208227 - 4{,}440 \times e^{-3.76\% \times 1} \times 0.123414 \\ &\cong 460.17 \end{aligned}$$

c)

$$B_0 = 100\% \times e^{-5.40\% \times 1} + RV_0.$$

$$RV_1 = 80\% \times \begin{cases} \left[\dfrac{S_0 - S_1}{S_0}\right]^+ \Leftarrow \sup\limits_{u \in [0,1]} (S_u) < 5{,}000 \\ 0\% \Leftarrow else \end{cases}$$

$$= \frac{80\%}{S_0} \times \begin{cases} [S_0 - S_1]^+ \Leftarrow \sup\limits_{u \in [0,1]} (S_u) < 5{,}000 \\ 0\% \Leftarrow else \end{cases}$$

$$= \frac{80\%}{S_0} \times UO(-1)_1 (S; X = S_0 = 4{,}440; U = 5{,}000; R = 0; T = 1y)$$

Portanto, o cupão pode ser avaliado como sendo uma *up-and-out European put* sobre o FTSE100 e com rebate zero:

$$RV_0 = \frac{80\%}{S_0} \times UO(-1)_0 (S; X = S_0 = 4{,}440; U = 5{,}000; R = 0; T = 1y).$$

Via equação (116),

$$UO(-1)_0 (S; X = S_0 = 4{,}440; U = 5{,}000; R = 0; T = 1y)$$

$$= p_0 (S = 4{,}440; X = 4{,}440; T = 1y) - \left(\frac{5{,}000}{4{,}440}\right)^{\frac{2\mu}{(0.2)^2}} \times p_0 \left(S = \frac{(5{,}000)^2}{4{,}440}; X = 4{,}440; T = 1y \right).$$

Visto que:

$$\mu = 5.4\% - 3.76\% - \frac{(0.2)^2}{2} \cong -0.0036;$$

$$p_0\left(S = 4{,}440; X = 4{,}440; T = 1y\right) = 304.20;$$

e que uma opção financeira é uma função homogénea de grau 1 no *spot* e no *strike*,

$$p_0\left(S = \frac{(5{,}000)^2}{4{,}440}; X = 4{,}440; T = 1y\right)$$

$$= \frac{(5{,}000)^2}{(4{,}440)^2} \times p_0\left(S = 4{,}440; X = \frac{(4{,}440)^3}{(5{,}000)^2} \cong 3{,}501.14; T = 1y\right)$$

$$= \frac{(5{,}000)^2}{(4{,}440)^2} \times 36.43 \cong 46.20$$

Então:

$$UO(-1)_0\left(S; X = S_0 = 4{,}440; U = 5{,}000; R = 0; T = 1y\right)$$

$$= 304.20 - \left(\frac{5{,}000}{4{,}440}\right)^{\frac{2\times(-0.0036)}{(0.2)^2}} \times 46.20$$

$$\cong 258.98$$

Em suma,

$$RV_0 = \frac{80\%}{4{,}440} \times 258.98 \cong 4.67\%$$

e

$$B_0 = 94.74\% + 4.67\% = 99.41\% < 100\% \Rightarrow \text{ Não comprar.}$$

d)

$$B_0 = 100\% \times e^{-5.40\%\times 1} + RV_0.$$

$$RV_1 = \begin{cases} \dfrac{S_1 - S_0}{S_0} \Leftarrow \inf_{u\in[0,1]}(S_u) > 3{,}552 \\ 0\% \Leftarrow else \end{cases}$$

$$= \frac{S_1}{S_0} \times 1_{\left\{\inf_{u\in[0,1]}(S_u) > 3{,}552\right\}} - 1_{\left\{\inf_{u\in[0,1]}(S_u) > 3{,}552\right\}}$$

Portanto,

$$RV_0 = \frac{e^{-5.40\%\times 1}}{S_0} \times E_0^Q\left(S_1 \times 1_{\left\{\inf_{u\in[0,1]}(S_u) > 3{,}552\right\}}\right) - e^{-5.40\%\times 1} \times \Pr_0^Q\left(\inf_{u\in[0,1]}(S_u) > 3{,}552\right).$$

Utilizando a equação (122), com $\eta = -1$,

$$\Pr_0^Q\left(\inf_{u\in[0,1]}(S_u) > 3{,}552\right)$$

$$= \Pr_0^Q\left\{\inf_{u\in[0,1]}\left[\ln\left(\frac{S_u}{4{,}440}\right)\right] > \ln\left(\frac{3{,}552}{4{,}440}\right)\right\}$$

$$= \Phi\left[d_2^M(4{,}440;3{,}552)\right] - \left(\frac{3{,}552}{4{,}440}\right)^{\frac{2(-0.0036)}{(0.2)^2}} \times \Phi\left[d_2^M(3{,}552;4{,}440)\right]$$

sendo:

$$d_2^M(4{,}440;3{,}552) = \frac{\ln\left(\frac{4{,}440}{3{,}552}\right) + \left[5.4\% - 3.76\% - \frac{(0.2)^2}{2}\right] \times 1}{0.2 \times \sqrt{1}} \cong 1.0977, \text{ e}$$

$$d_2^M(3{,}552;4{,}440) = \frac{\ln\left(\frac{3{,}552}{4{,}440}\right) + \left[5.4\% - 3.76\% - \frac{(0.2)^2}{2}\right] \times 1}{0.2 \times \sqrt{1}} \cong -1.1337.$$

Via tabela,

$$\Phi\left[d_2^M(4{,}440;3{,}552)\right] \approx \Phi(1.1) == 0.8643.$$

$$\Phi\left[d_2^M(3{,}552;4{,}440)\right] \approx \Phi(-1.13) = 1 - \Phi(1.13) = 1 - 0.8708 = 0.1292.$$

Portanto,

$$\Pr_0^Q\left(\inf_{u\in[0,1]}(S_u) > 3{,}552\right) = 0.8643 - \left(\frac{3{,}552}{4{,}440}\right)^{\frac{2(-0.0036)}{(0.2)^2}} \times 0.1292 \cong 0.7298.$$

Relativamente ao valor esperado, utilizando a fdp (98):

$$E_0^Q\left(S_1 \times 1_{\left\{\inf_{u\in[0,1]}(S_u)>3{,}552\right\}}\right)$$

$$= \int_{-\infty}^{+\infty} S_0 e^y \times 1_{\left\{y>\ln\left(\frac{3{,}552}{4{,}440}\right)\right\}} \times \left\{\phi[y;-0.0036,20\%] - \left(\frac{3{,}552}{4{,}440}\right)^{\frac{2(-0.0036)}{(0.2)^2}} \times \phi\left[y;2\ln\left(\frac{3{,}552}{4{,}440}\right) - 0.0036,20\%\right]\right\} dy$$

$$\Leftrightarrow E_0^Q\left(S_1 \times 1_{\left\{\inf_{u\in[0,1]}(S_u)>3,552\right\}}\right)$$

$$= 4,440\times\int_{\ln\left(\frac{3,552}{4,440}\right)}^{+\infty} e^y \frac{1}{0.2\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{y+0.0036}{0.2}\right)^2\right]dy - 4,621.97\times\int_{\ln\left(\frac{3,552}{4,440}\right)}^{+\infty} e^y \frac{1}{0.2\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{y+0.4499}{0.2}\right)^2\right]dy$$

$$= 4,440\times\int_{\ln\left(\frac{3,552}{4,440}\right)}^{+\infty} \frac{1}{0.2\sqrt{2\pi}}\exp\left[-\frac{1}{2}\frac{y^2-2\times(0.04-0.0036)y+(0.0036)^2}{0.04}\right]dy$$
$$-4,621.97\times\int_{\ln\left(\frac{3,552}{4,440}\right)}^{+\infty} \frac{1}{0.2\sqrt{2\pi}}\exp\left[-\frac{1}{2}\frac{y^2+2\times(0.4499-0.04)y+(0.4499)^2}{0.04}\right]dy$$

$$= 4,440\times\exp\left[-\frac{1}{2}\frac{(0.0036)^2-(0.0364)^2}{0.04}\right]\int_{\ln\left(\frac{3,552}{4,440}\right)}^{+\infty} \frac{1}{0.2\sqrt{2\pi}}\exp\left[-\frac{1}{2}\frac{y^2-2\times 0.0364y+(0.0364)^2}{0.04}\right]dy$$
$$-4,621.97\times\exp\left[-\frac{1}{2}\frac{(0.4499)^2-(0.4099)^2}{0.04}\right]\int_{\ln\left(\frac{3,552}{4,440}\right)}^{+\infty} \frac{1}{0.2\sqrt{2\pi}}\exp\left[-\frac{1}{2}\frac{y^2+2\times 0.4099y+(0.4099)^2}{0.04}\right]dy$$

$$= 4,513.42\times\Phi\left[-\frac{\ln\left(\frac{3,552}{4,440}\right)-0.0364}{0.2}\right]-3,006.93\times\Phi\left[-\frac{\ln\left(\frac{3,552}{4,440}\right)+0.4099}{0.2}\right]$$

$$\cong 4,513.42\times\Phi[1.30]-3,006.93\times\Phi[-0.93]$$

$$= 4,513.42\times 0.9032-3,006.93\times(1-0.8238)\cong 3,546.70$$

Em suma,

$$RV_0 = \frac{e^{-5.40\%\times 1}}{S_0}\times E_0^Q\left(S_1\times 1_{\left\{\inf_{u\in[0,1]}(S_u)>3,552\right\}}\right) - e^{-5.40\%\times 1}\times \Pr_0^Q\left(\inf_{u\in[0,1]}(S_u)>3,552\right).$$
$$= \frac{e^{-5.40\%\times 1}}{4,440}\times 3,546.70 - e^{-5.40\%\times 1}\times 0.7298 \cong 6.54\%$$

$$B_0 = 94.74\% + 6.54\% = 101.28\% > 100\% \Rightarrow \text{ Depositar.}$$