Modelos de Estrutura Temporal de Taxas de Juro
Mestrado em Matemática Financeira 17/18
IBS e FCUL
Exame 1ª Época - Resolução

17/Dec/19 Duration: 3h

1. (a) The Euribor rate $E(T_1,T_2)$ can be written in terms of zero-coupon bond prices, since

$$P(T_1,T_2) = [1 + (T_2 - T_1) \times E(T_1,T_2)]^{-1}$$

implies that

$$\begin{aligned} E(T_1,T_2) &= \frac{1}{T_2 - T_1}\left[\frac{1}{P(T_1,T_2)} - 1\right] \\ &= \frac{1}{T_2 - T_1}\left[\frac{P(T_1,T_1)}{P(T_1,T_2)} - 1\right] \\ &= \frac{1}{T_2 - T_1}\left[\frac{1}{P(T_1,T_1,T_2)} - 1\right], \end{aligned} \tag{1}$$

where $P(t,T_1,T_2) = \frac{P(t,T_2)}{P(t,T_1)}$ represents the time-t forward price for delivery at time T_1 ($\geq t$) of a zero-coupon bond with maturity at time T_2.
Using equation (368) of the handouts, then

$$P(T_1,T_1,T_2) = \frac{P(t,T_2)}{P(t,T_1)} \exp\left[\frac{1}{2}v^2(t,T_1,T_2) - Y^{\mathbb{Q}_2}\right], \tag{2}$$

where

$$Y^{\mathbb{Q}_2} := \rho \int_t^{T_1} [B(T_2 - s) - B(T_1 - s)]\, dW_s^{\mathbb{Q}_2} \Big| \mathcal{F}_t \overset{\mathbb{Q}_2}{\sim} N^1\left(0, v^2(t,T_1,T_2)\right).$$

Combining equations (1) and (2), then

$$E(T_1,T_2) = \frac{1}{T_2 - T_1}\left\{\frac{P(t,T_1)}{P(t,T_2)} \exp\left[-\frac{1}{2}v^2(t,T_1,T_2) + Y^{\mathbb{Q}_2}\right] - 1\right\}. \tag{3}$$

Therefore, and for $t \leq T_1$,

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}_2}[E(T_1,T_2)|\mathcal{F}_t] \\ = \;& \frac{1}{T_2 - T_1}\left\{\frac{P(t,T_1)}{P(t,T_2)} \exp\left(-\frac{1}{2}v^2(t,T_1,T_2)\right) \mathbb{E}_{\mathbb{Q}_2}\left[\exp\left(Y^{\mathbb{Q}_2}\right)|\mathcal{F}_t\right] - 1\right\}, \end{aligned} \tag{4}$$

while

$$\mathbb{E}_{\mathbb{Q}_2}\left[\exp\left(Y^{\mathbb{Q}_2}\right)|\mathcal{F}_t\right] = \exp\left(0 + \frac{1}{2}v^2(t,T_1,T_2)\right) \tag{5}$$

is the moment generating function of a normal random variable.

Finally, combining equations (4) and (5), then

$$\begin{aligned}
& \mathbb{E}_{\mathbb{Q}_2}\left[E\left(T_1,T_2\right)|\mathcal{F}_t\right] \\
= & \frac{1}{T_2-T_1}\left\{\frac{P\left(t,T_1\right)}{P\left(t,T_2\right)}\exp\left(-\frac{1}{2}v^2\left(t,T_1,T_2\right)\right)\exp\left(0+\frac{1}{2}v^2\left(t,T_1,T_2\right)\right)-1\right\} \\
= & \frac{1}{T_2-T_1}\left[\frac{P\left(t,T_1\right)}{P\left(t,T_2\right)}-1\right] \\
= & E\left(t,T_1,T_2\right).
\end{aligned}$$

(b) Using Proposition 81 of the handouts, and for $\lambda,\mu\in\mathbb{R}_+$,

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-\lambda r_t}\exp\left(-\mu\int_{t_0}^{t}r_s ds\right)|\mathcal{F}_{t_0}\right]=\exp\left[\phi_{\lambda,\mu}\left(t-t_0\right)-r_{t_0}\psi_{\lambda,\mu}\left(t-t_0\right)\right],\tag{6}$$

where

$$\phi_{\lambda,\mu}\left(t-t_0\right):=\frac{2k\theta}{\sigma^2}\ln\left[\frac{2he^{\frac{(k+h)(t-t_0)}{2}}}{\sigma^2\lambda\left(e^{h(t-t_0)}-1\right)+h-k+\left(k+h\right)e^{h(t-t_0)}}\right],\tag{7}$$

and

$$\psi_{\lambda,\mu}\left(t-t_0\right):=\frac{\lambda\left[h+k+\left(h-k\right)e^{h(t-t_0)}\right]+2\mu\left[e^{h(t-t_0)}-1\right]}{\sigma^2\lambda\left(e^{h(t-t_0)}-1\right)+h-k+\left(k+h\right)e^{h(t-t_0)}},\tag{8}$$

with

$$h:=\sqrt{k^2+2\sigma^2\mu}.\tag{9}$$

Therefore, the moment generating function (with parameter ω) of the random variable

$$y_t:=\int_{t_0}^{t}r_s ds$$

is given by

$$\begin{aligned}
M\left(\omega\right) &= \mathbb{E}_{\mathbb{Q}}\left[\exp\left(\omega y_t\right)|\mathcal{F}_{t_0}\right] \\
&= \exp\left[\phi_{0,-\omega}\left(t-t_0\right)-r_{t_0}\psi_{0,-\omega}\left(t-t_0\right)\right].
\end{aligned}\tag{10}$$

On the other hand, we also know that

$$VAR\left(y_t|\mathcal{F}_{t_0}\right)=\mathbb{E}_{\mathbb{Q}}\left(y_t^2|\mathcal{F}_{t_0}\right)-\left[\mathbb{E}_{\mathbb{Q}}\left(y_t|\mathcal{F}_{t_0}\right)\right]^2.\tag{11}$$

Through the differentiation of the moment generating function (10) we can obtain the moments contained on the right-hand side of equation (11):

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left(y_t|\mathcal{F}_{t_0}\right) &= \left.\frac{\partial M\left(\omega\right)}{\partial\omega}\right|_{\omega=0} \\
&= \left[\left.\frac{\partial\phi_{0,-\omega}\left(t-t_0\right)}{\partial\omega}\right|_{\omega=0}-r_{t_0}\left.\frac{\partial\psi_{0,-\omega}\left(t-t_0\right)}{\partial\omega}\right|_{\omega=0}\right],
\end{aligned}\tag{12}$$

and

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left(y_t^2|\mathcal{F}_{t_0}\right) &= \left.\frac{\partial^2 M(\omega)}{\partial\omega^2}\right|_{\omega=0} \\
&= \left[\left.\frac{\partial^2\phi_{0,-\omega}(t-t_0)}{\partial\omega^2}\right|_{\omega=0} - r_{t_0}\left.\frac{\partial^2\psi_{0,-\omega}(t-t_0)}{\partial\omega^2}\right|_{\omega=0}\right]. \qquad (13)
\end{aligned}$$

(c) The terminal payoff of a *cash-or-nothing put* with expiry date at time T, with a *contract size* equal to M, with a *strike* equal to k, and on the Euribor rate $E(T,T+\delta)$ to prevail between times T ($\geq t$) and $T+\delta$ (with $\delta>0$) is equal to:

$$V_T = M\mathbb{1}_{\{E(T,T+\delta)<k\}}. \qquad (14)$$

Using the identity

$$P(T,T+\delta) = [1+\delta\times E(T,T+\delta)]^{-1},$$

then equation (14) can be rewritten as

$$\begin{aligned}
V_T &= M\mathbb{1}_{\{P^{-1}(T,T+\delta)-1<k\delta\}} \\
&= M\mathbb{1}_{\{P^{-1}(T,T+\delta)<1+k\delta\}} \\
&= M\mathbb{1}_{\left\{P(T,T+\delta)>(1+k\delta)^{-1}\right\}}.
\end{aligned}$$

Consequently, and using the *forward measure* $\mathbb{Q}_T$ associated to the numeraire $P(t,T)$, then

$$\begin{aligned}
V_t &= P(t,T)\,\mathbb{E}_{\mathbb{Q}_T}\left(\left.\frac{M\mathbb{1}_{\left\{P(T,T+\delta)>(1+k\delta)^{-1}\right\}}}{P(T,T)}\right|\mathcal{F}_t\right) \\
&= MP(t,T)\,\mathbb{Q}_T\left[P(T,T+\delta)>(1+k\delta)^{-1}\right|\left.\mathcal{F}_t\right]. \qquad (15)
\end{aligned}$$

Using equation (389) of the handouts, then equation (15) yields

$$\begin{aligned}
V_t &= MP(t,T)\,\mathbb{Q}_T\left[\exp\left(A(\delta)+B(\delta)\,r_T\right)>(1+k\delta)^{-1}\right|\left.\mathcal{F}_t\right] \\
&= MP(t,T)\,\mathbb{Q}_T\left[r_T<\left.\frac{\ln(1+k\delta)^{-1}-A(\delta)}{B(\delta)}\right|\mathcal{F}_t\right], \qquad (16)
\end{aligned}$$

since $B(\delta)<0$.
Equation (425), (426) and (441) of the handouts show that

$$\mathbb{E}_{\mathbb{Q}_T}\left[\exp\left(-\lambda\frac{r_T}{L_T}\right)|\mathcal{F}_t\right] = \frac{\exp\left(-\frac{\lambda}{1+2\lambda}\zeta_T\right)}{(1+2\lambda)^{\frac{2k\theta}{\sigma^2}}}, \qquad (17)$$

where

$$\zeta_T := \frac{8r_t\gamma^2e^{\gamma(T-t)}}{\sigma^2\left[e^{\gamma(T-t)}-1\right]\left\{\gamma\left[e^{\gamma(T-t)}+1\right]+k\left[e^{\gamma(T-t)}-1\right]\right\}}, \qquad (18)$$

and

$$L_T := \frac{\sigma^2}{2} \frac{e^{\gamma(T-t)} - 1}{\gamma \left[e^{\gamma(T-t)} + 1\right] + k \left[e^{\gamma(T-t)} - 1\right]}. \tag{19}$$

Therefore, equation (16) becomes

$$\begin{aligned} V_t &= MP(t,T)\, \mathbb{Q}_T \left[\frac{r_T}{L_T} < \frac{\ln (1+k\delta)^{-1} - A(\delta)}{B(\delta)\, L_T} \middle| \mathcal{F}_t \right] \\ &= MP(t,T)\, F_{\chi^2_{\left(\frac{4k\theta}{\sigma^2}\varsigma_{T+\delta}\right)}} \left(\frac{\ln (1+k\delta)^{-1} - A(\delta)}{B(\delta)\, L_T} \right). \end{aligned}$$

(a) Using equation (53) of the handouts, the fair value of the European-style call (with $\beta < 2$) is given by:

$$p_t(S_t, X, T) = Xe^{-r(T-t)} Q_{\chi^2\left(\frac{2}{2-\beta}, 2\kappa X^{2-\beta}\right)}(2x) - S_t e^{-q(T-t)} \left[1 - Q_{\chi^2\left(2+\frac{2}{2-\beta}, 2x\right)}\left(2\kappa X^{2-\beta}\right)\right], \tag{20}$$

where

$$\kappa := \frac{2(r-q)}{(2-\beta)\,\delta^2 \left[e^{(2-\beta)(r-q)(T-t)} - 1\right]}, \tag{21}$$

and

$$x := \kappa S_t^{2-\beta} e^{(2-\beta)(r-q)(T-t)}. \tag{22}$$

Since the (annualized) standard deviation of stock returns is equal to 20% per year, then, and using equation (2) of the handouts,

$$\delta = \frac{20\%}{(10)^{\frac{1-2}{2}}} = 0.632456.$$

Using equations (21) and (22),

$$\kappa = \frac{2(1\% - 2\%)}{(2-1)(0.632456)^2 \left[e^{(2-1)(1\%-2\%)\times 0.25} - 1\right]} \cong 20.02501042,$$

and

$$x = (20.02501042) \times (10)^{2-1} e^{(2-1)(1\%-2\%)\times 0.25} \cong 199.7501042.$$

Hence, equation (20) yields

$$\begin{aligned} p_t &= 10 \times e^{-1\%\times 0.25} \times Q_{\chi^2\left(\frac{2}{2-1}, 2\times 20.02501042\times 10^{2-1}\right)}(2 \times 199.7501042) \\ &\quad -10 \times e^{-2\%\times 0.25} \times F_{\chi^2\left(2+\frac{2}{2-1}, 2\times 199.7501042\right)}\left(2 \times 20.02501042 \times 10^{2-1}\right) \\ &= 10 \times e^{-1\%\times 0.25} \times Q_{\chi^2(2, 400.5002083)}(399.5002083) \\ &\quad -10 \times e^{-2\%\times 0.25} \times F_{\chi^2(4, 399.5002083)}(400.5002083). \end{aligned} \tag{23}$$

From the table provided in the exam, we know that

$$Q_{\chi^2(2, 400.5002083)}(399.5002083) = 1 - 0.48006 = 0.51994. \tag{24}$$

The probability $F_{\chi^2(4,399.5002083)}(400.5002083)$ can be computed using the Sankaran approximation, i.e.

$$\begin{aligned} F_{\chi^2(a,b)}(z) &= \mathbb{Q}\left(\chi^2(a,b) < z\right) \\ &= \mathbb{Q}\left\{\left[\frac{\chi^2(a,b)}{a+b}\right]^h < \left(\frac{z}{a+b}\right)^h\right\} \\ &\approx \Phi\left[\frac{\left(\frac{z}{a+b}\right)^h - \mu_h}{\sigma_h}\right], \end{aligned} \tag{25}$$

where

$$\mu_h := 1 + h(h-1)\frac{a+2b}{(a+b)^2} - h(h-1)(2-h)(1-3h)\frac{(a+2b)^2}{2(a+b)^4}, \tag{26}$$

$$\sigma_h^2 := h^2\frac{2(a+2b)}{(a+b)^2}\left[1 - (1-h)(1-3h)\frac{a+2b}{(a+b)^2}\right], \tag{27}$$

and

$$h := 1 - \frac{2}{3}(a+b)(a+3b)(a+2b)^{-2}. \tag{28}$$

Since $a = 4$ and $b = 399.5002083$, then

$$\begin{aligned} h &= 1 - \frac{2}{3}(4+399.5002083)(4+3\times 399.5002083) \\ &\quad (4+2\times 399.5002083)^{-2} \\ &\cong 0.498343696, \end{aligned}$$

$$\begin{aligned} \mu_h &= 1 + 0.498343696\times(0.498343696-1)\frac{4+2\times 399.5002083}{(4+399.5002083)^2} \\ &\quad -0.498343696\times(0.498343696-1)(2-0.498343696) \\ &\quad (1-3\times 0.498343696)\frac{(4+2\times 399.5002083)^2}{2(4+399.5002083)^4} \\ &\cong 0.998764739, \end{aligned}$$

and

$$\begin{aligned} \sigma_h^2 &= 0.498343696^2 \times \frac{2(4+2\times 399.5002083)}{(4+399.5002083)^2} \\ &\quad \left[1-(1-0.498343696)(1-3\times 0.498343696)\frac{4+2\times 399.5002083}{(4+399.5002083)^2}\right] \\ &\cong 0.002452719. \end{aligned}$$

Using equation (25), then

$$\begin{aligned} & F_{\chi^2(4,399.5002083)}(400.5002083) \\ &= \Phi\left[\frac{\left(\frac{400.5002083}{4+399.5002083}\right)^{0.498343696} - 0.998764739}{\sqrt{0.002452719}}\right] \\ &\cong 0.480056502. \end{aligned} \tag{29}$$

Finally, combining equations (23), (24) and (29), then

$$\begin{aligned} p_t &= 10 \times e^{-1\% \times 0.25} \times 0.51994 - 10 \times e^{-2\% \times 0.25} \times 0.480056502 \\ &\cong EUR0.40975. \end{aligned}$$

(b) The terminal payoff of this contract is equal to

$$V_T = EUR10 \times \mathbb{1}_{\left\{\tau_0^S \leq T\right\}}.$$

Therefore, the value today of this (credit risk) contract is simply the value of a European-style put option conditional on no default until time T:

$$\begin{aligned} V_t &= EUR10 \times e^{-r(T-t)} \times \mathbb{E}_{\mathbb{Q}} \left(\mathbb{1}_{\left\{\tau_0^S \leq T\right\}} \middle| \mathcal{F}_t \right) \\ &= EUR10 \times e^{-r(T-t)} \times \mathbb{Q} \left(\tau_0^S \leq T \middle| \mathcal{F}_t \right) \\ &= p_t^D \left(S_t, K = EUR10, T \right) \\ &= EUR10 \times e^{-1\% \times 0.25} \times \frac{\Gamma \left(\frac{2-\gamma}{2}, \frac{S_t^{\frac{2}{2-\gamma}}}{2f(\tau)} \right)}{\Gamma \left(1 - \frac{\gamma}{2} \right)} \\ &= EUR10 \times e^{-1\% \times 0.25} \times Q_{\chi^2(2-\gamma,0)} \left(\frac{S_t^{\frac{2}{2-\gamma}}}{f(\tau)} \right), \end{aligned} \tag{30}$$

where the last two lines follow from equations (32) and (51) of the handouts,

$$\begin{aligned} \gamma &= \frac{2 - 2\beta}{2 - \beta} \\ &= \frac{2 - 2 \times 1}{2 - 1} \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} f(\tau) &= \frac{\delta^2}{2(r-q)(2-\gamma)} \left[1 - \exp \left(-\frac{2(r-q)\tau}{(2-\gamma)} \right) \right] \\ &= \frac{0.632456^2}{2(1\% - 2\%)(2-0)} \left[1 - \exp \left(-\frac{2(1\% - 2\%) \times 0.25}{(2-0)} \right) \right] \\ &\cong 0.025031276. \end{aligned}$$

Using the table provided in the exam, then

$$\begin{aligned} Q_{\chi^2(2-\gamma,0)} \left(\frac{S_t^{\frac{2}{2-\gamma}}}{f(\tau)} \right) &= Q_{\chi^2(2-0,0)} \left(\frac{10^{\frac{2}{2-0}}}{0.025031276} \right) \\ &= Q_{\chi^2(2,0)} (399.5002083) \\ &= 1 - 1 \\ &= 0, \end{aligned}$$

and equation (30) yields:

$$\begin{aligned} V_t &= EUR10 \times e^{-1\% \times 0.25} \times 0 \\ &= 0. \end{aligned}$$

(a) Using Proposition 37 of the handouts,

$$\begin{aligned} c_0 &= 10 \times e^{-2\% \times 0.25} \times P_1 (S_t = 10, v_t = 0.04; T = 0.25, X = 12) \qquad (31) \\ & -e^{-3\% \times 0.5} \times 12 \times P_2 (S_t = 10, v_t = 0.04; T = 0.25, X = 12) \,. \end{aligned}$$

Using equations (220) and (221) of the handouts:

$$\begin{aligned} P_1 (S_t = 10, v_t = 0.04; T = 0.25, X = 12) &\approx \frac{1}{2} + \frac{-1.40646658}{\pi} \\ &\cong 0.052308, \qquad (32) \end{aligned}$$

and

$$\begin{aligned} P_2 (S_t = 10, v_t = 0.04; T = 0.25, X = 12) &\approx \frac{1}{2} + \frac{-1.43926511}{\pi} \\ &\cong 0.041868. \qquad (33) \end{aligned}$$

Combining equations (31), (32) and (33), then

$$\begin{aligned} c_0 &= 10 \times e^{-2\% \times 0.25} \times 0.052308 - e^{-3\% \times 0.5} \times 12 \times 0.041868 \\ &\cong EUR0.02181. \end{aligned}$$

(b) The fair value of the ATM European-style cash-or-nothing call is:

$$V_t = EUR1 \times e^{-3\% \times 0.25} \times \mathbb{Q} \left(S_T > 10 | \mathcal{F}_t \right), \qquad (34)$$

with $t = 0$ and $T = 0.25$.
Using the table provided in the exam, then

$$\begin{aligned} \mathbb{Q} \left(S_T > 10 | \mathcal{F}_t \right) &= P_2 (S_t = 10, v_t = 0.04; T = 0.25, X = 10) \\ &\approx \frac{1}{2} + \frac{-0.02401091}{\pi} \\ &\cong 0.492357. \qquad (35) \end{aligned}$$

Combining equations (34) and (35), we finally get

$$\begin{aligned} V_t &= EUR1 \times e^{-3\% \times 0.25} \times 0.492357 \\ &\cong EUR0.4887. \end{aligned}$$

(a) The purpose is to price a bond with the following future cash flows:
Therefore,

$$\begin{aligned} B_0 &= 0.15\% \times P(0, 0.3142) + 0.15\% \times P(0, 1.3142) + 100.15\% \times P(0, 2.3142) \\ &= 0.15\% \times P(0, 0.3142) + 0.15\% \times 0.9887 + 100.15\% \times 0.9790. \qquad (36) \end{aligned}$$

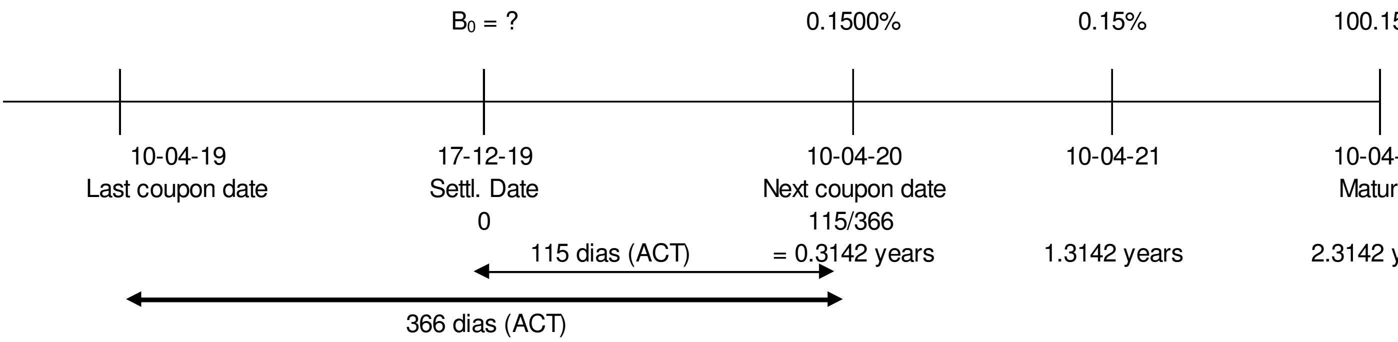

Figure 1:

Concerning the discount factor for the maturity of 0.3142 years, equations (316) and (317) of the handouts imply that

$$\begin{aligned} B(0.3142) &= \frac{1-e^{-3\times 0.3142}}{3} \\ &\cong 0.2035, \end{aligned}$$

and

$$\begin{aligned} A(0.3142) &= (0.2035-0.3142)\left(1\%-\frac{0.05^2}{2(3)^2}\right)-\frac{0.05^2}{4\times 3}\times(0.2035)^2 \\ &\cong -0.0011. \end{aligned}$$

Hence,

$$\begin{aligned} P(0,0.3142) &= \exp(-0.0011-0.2035\times 0.5\%) \\ &\cong 0.9979. \end{aligned}$$

Recalling equation (36), then

$$\begin{aligned} B_0 &= 0.15\%\times 0.9979+0.15\%\times 0.9887+100.15\%\times 0.9790 \\ &\cong 98.3447\%. \end{aligned}$$

(b) On 17/12/19, the accrued interest amount is equal to

$$AI_0 = 0.15\%\times\frac{366-115}{366}\cong 0.103\%.$$

On the other hand, the ask gross price is:

$$\begin{aligned} GP_0^{ask} &= \frac{0.15\%}{(1+0.859\%)^{0.3142}}+\frac{0.15\%}{(1+0.859\%)^{1.3142}}+\frac{100.15\%}{(1+0.859\%)^{2.3142}} \\ &\cong 98.4029\%. \end{aligned}$$

(c) Using Proposition 74 of the handouts,

$$\begin{aligned}
& c_0\left[P(0,1.3142);K=99.170\%;0.3142\right] \\
= \; & P(0,1.3142)\times\Phi\left(d_1^V\right)-0.9917\times P(0,0.3142)\times\Phi\left(d_0^V\right) \\
= \; & 0.9887\times\Phi\left(d_1^V\right)-0.9917\times 0.9979\times\Phi\left(d_0^V\right),
\end{aligned}$$

where

$$\begin{aligned}
v(0,0.3142,1.3142) &= \sqrt{\frac{0.05^2}{3^2}\left[1-e^{-3\times 1}\right]^2\frac{1-e^{-2\times 3\times 0.3142}}{2\times 3}} \\
&\cong 0.595\%,
\end{aligned}$$

$$\begin{aligned}
d_1^V &= \frac{\ln\left(\frac{0.9887}{0.9917\times 0.9979}\right)+\frac{(0.595\%)^2}{2}}{0.595\%} \\
&\cong -0.154467226,
\end{aligned}$$

and

$$\begin{aligned}
d_0^V &= -0.154467226-0.595\% \\
&\cong -0.160421726.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& c_0\left[P(0,1.3142);K=99.170\%;0.3142\right] \\
= \; & 0.9887\times\Phi(-0.154467226)-0.9917\times 0.9979\times\Phi(-0.160421726) \\
= \; & 0.9887\times 0.438620676-0.9917\times 0.9979\times 0.436274438 \\
\cong \; & 0.192\%. \qquad (37)
\end{aligned}$$

(d) Using Proposition 76 of the handouts, the fair value of a European-style *call* on a CBB can be decomposed into a portfolio of 2 European-style *calls* on PBDs:

$$\begin{aligned}
& c_0(B_0;X=98.50\%;T=0.3142) \qquad (38) \\
= \; & 0.15\%\times c_0\left[P(0,1.3142);X_1;T=0.3142\right] \\
& +100.15\%\times c_0\left[P(0,2.3142);X_2;T=0.3142\right].
\end{aligned}$$

The *strikes* can be obtained through equation (374) of the handouts:

$$\begin{aligned}
X_1 &= \exp\left[A(1.3142-0.3142)-B(1)\times 0.5\%\right] \\
&= \exp(-0.0068-0.3167\times 0.5\%) \\
&\cong 99.170\%,
\end{aligned}$$

and

$$\begin{aligned}
X_2 &= \exp\left[A(2.3142-0.3142)-B(2)\times 0.5\%\right] \\
&= \exp(-0.0165-0.3325\times 0.5\%) \\
&\cong 98.204\%.
\end{aligned}$$

Hence,

$$\begin{aligned} & c_0\left(B_0; X=98.50\%; T=0.3142\right) \quad (39)\\ = & \ 0.15\% \times c_0\left[P(0,1.3142); X_1=99.170\%; T=0.3142\right] \\ & +100.15\% \times c_0\left[P(0,2.3142); X_2=98.204\%; T=0.3142\right]. \end{aligned}$$

The first *call* was already priced in the previous question—please see equation (37):

$$c_0\left[P(0,1.3142); X_1=99.170\%; T=0.3142\right] \cong 0.192\%. \quad (40)$$

Concerning the second *call*, the exam provides the following market price:

$$c_0\left[P(0,2.3142); X_2=98.204\%; T=0.3142\right] = 0.199\%. \quad (41)$$

Combining equations (39), (40) and (41),

$$\begin{aligned} & c_0\left(B_0; X=98.50\%; T=0.3142\right) \\ = & \ 0.15\% \times 0.192\% + 100.15\% \times 0.199\% \\ \cong & \ 0.200\%. \end{aligned}$$

(a) Using equation (239) of the handouts, the fixed rate for a 2-years IRS with annual revolving:

$$\begin{aligned} IRS &= \frac{1-P(0,2)}{1\times\left[P(0,1)+P(0,2)\right]} \\ &= \frac{1-0.95109026}{0.97888160+0.95109026} \\ &\cong 2.534\%. \end{aligned}$$

(b) The purpose is to price the following option contract:

$$p_0\left[P(0,2); K=97\%; 1\right].$$

Using Proposition 83 of the handouts,

$$\begin{aligned} & p_0\left[P(0,2); K=97\%; 1\right] \quad (42)\\ = & -P(0,2)\times Q_{\chi^2_{\left(\frac{4\times 2\times 3\%}{0.1^2},\zeta_2\right)}}\left(\frac{r^*}{L_2}\right) \\ & +0.97\times P(0,1)\times F_{\chi^2_{(24,\zeta_1)}}\left(\frac{r^*}{L_1}\right), \end{aligned}$$

where $\gamma \cong 2.005$,

$$\begin{aligned} \zeta_1 &= \frac{8r_t\gamma^2 e^{\gamma(T_1-t)}}{\sigma^2\left[e^{\gamma(T_1-t)}-1\right]\left\{\gamma\left[e^{\gamma(T_1-t)}+1\right]+k\left[e^{\gamma(T_1-t)}-1\right]\right\}} \\ &= \left[8\times 1\%\times(2.005)^2\times e^{2.005\times 1}\right]\left\{0.1^2\times\left(e^{2.005\times 1}-1\right)\right. \\ & \quad \left.\left[2.005\times\left(e^{2.005\times 1}+1\right)+2\times\left(e^{2.005\times 1}-1\right)\right]\right\}^{-1} \\ &\cong 1.249388035, \end{aligned}$$

$$
\begin{aligned}
L_1 &= \frac{\sigma^2}{2} \frac{e^{\gamma(T_1-t)}-1}{\gamma\left[e^{\gamma(T_1-t)}+1\right]+k\left[e^{\gamma(T_1-t)}-1\right]} \\
&= \frac{\frac{0.1^2}{2}\times\left(e^{2.005\times 1}-1\right)}{2.005\times\left(e^{2.005\times 1}+1\right)+2\times\left(e^{2.005\times 1}-1\right)} \\
&\cong 0.001080143,
\end{aligned}
$$

and

$$
\begin{aligned}
r^* &= \frac{\ln(K)-A(T_2-T_1)}{B(T_2-T_1)} \\
&= \frac{\ln(0.97)-A(2-1)}{B(1)} \\
&= \frac{\ln(0.97)-(-0.01702401)}{-0.43205740} \\
&\cong 3.110\%.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& p_0\left[P(0,2);K=97\%;1\right] \\
= & -0.95109026\times Q_{\chi^2_{(24,1.248222984)}}\left(\frac{3.110\%}{0.001079136}\cong 28.81550841\right) \\
& +0.97\times 0.97888160\times F_{\chi^2_{(24,1.249388035)}}\left(\frac{3.110\%}{0.00108014}\cong 28.78863802\right).
\end{aligned} \tag{43}
$$

Using the table provided in the exam, we can compute the two probabilities contained in the previous equation:

$$
Q_{\chi^2_{(24,1.248222984)}}(28.81550841) = 1-0.71380513 = 0.28619487, \tag{44}
$$

and

$$
F_{\chi^2_{(24,1.249388035)}}(28.78863802) = 0.71259693. \tag{45}
$$

Finally, combining equations (43), (44) and (45), then

$$
\begin{aligned}
& p_0\left[P(0,2);K=97\%;1\right] \\
= & -0.95109026\times 0.28619487+0.97\times 0.97888160\times 0.71259693 \\
\cong & 0.00069641.
\end{aligned}
$$