Modelos de Estrutura Temporal de Taxas de Juro
Mestrado em Matemática Financeira 12/13
IBS e FCUL
Exame 1ª Época - Resolução

19/Dec/13 Duration: 3h

1. (a) Based on equations (29) and (59) of the handouts,

$$\begin{aligned} p_t(S_t, X, T) &= -S_t e^{-q(T-t)} F_{\chi^2\left(2+\frac{2}{2-\beta}, 2x(S_t;\tau)\right)}\left(2\kappa(\tau) X^{2-\beta}\right) \\ &\quad + X e^{-r(T-t)} Q_{\chi^2\left(\frac{2}{2-\beta}, 2\kappa(\tau) X^{2-\beta}\right)}\left(2x(S_t;\tau)\right), \end{aligned} \tag{1}$$

where

$$\kappa(\tau) := \frac{2(r-q)}{(2-\beta)\delta^2\left[e^{(2-\beta)(r-q)\tau} - 1\right]}, \tag{2}$$

and

$$x(S;\tau) := \kappa(\tau) S^{2-\beta} e^{(2-\beta)(r-q)\tau}. \tag{3}$$

We want to show that

$$p_t(\lambda S_t, \lambda X, T) = \lambda p_t(S_t, X, T). \tag{4}$$

Using equation (1), then

$$\begin{aligned} p_t(\lambda S_t, \lambda X, T) &= -\lambda S_t e^{-q(T-t)} F_{\chi^2\left(2+\frac{2}{2-\beta}, 2x(\lambda S_t;\tau)\right)}\left(2\kappa(\tau)(\lambda X)^{2-\beta}\right) \\ &\quad + \lambda X e^{-r(T-t)} Q_{\chi^2\left(\frac{2}{2-\beta}, 2\kappa(\tau)(\lambda X)^{2-\beta}\right)}\left(2x(\lambda S_t;\tau)\right) \\ &= \lambda\Big[-S_t e^{-q(T-t)} F_{\chi^2\left(2+\frac{2}{2-\beta}, 2x(\lambda S_t;\tau)\right)}\left(2\kappa(\tau)(\lambda X)^{2-\beta}\right) \\ &\quad + X e^{-r(T-t)} Q_{\chi^2\left(\frac{2}{2-\beta}, 2\kappa(\tau)(\lambda X)^{2-\beta}\right)}\left(2x(\lambda S_t;\tau)\right)\Big], \end{aligned} \tag{5}$$

and equation (4) follows as long as both probabilities on the right-hand side of equation (5) are homogeneous of degree zero in the last two arguments, which is the case as long as

$$\sigma^2 = \delta^2 S_t^{\beta-2},$$

i.e

$$\delta^2 = \frac{\sigma^2}{S_t^{\beta-2}}, \tag{6}$$

where $\sigma \in \mathbb{R}^+$.

Combining equations (2) and (6), then

$$\begin{aligned}
2\kappa\left(\tau\right)\left(\lambda X\right)^{2-\beta} &= \lambda^{2-\beta}2\kappa\left(\tau\right)X^{2-\beta} \\
&= \lambda^{2-\beta}2\frac{2\left(r-q\right)}{\left(2-\beta\right)\frac{\sigma^2}{\lambda^{\beta-2}S_t^{\beta-2}}\left[e^{(2-\beta)(r-q)\tau}-1\right]}X^{2-\beta} \\
&= 2\frac{2\left(r-q\right)}{\left(2-\beta\right)\frac{\sigma^2}{S_t^{\beta-2}}\left[e^{(2-\beta)(r-q)\tau}-1\right]}X^{2-\beta} \\
&= 2\kappa\left(\tau\right)X^{2-\beta}. \qquad (7)
\end{aligned}$$

Similarly, combining equations (3) and (6), then

$$\begin{aligned}
2x\left(\lambda S_t;\tau\right) &= 2\kappa\left(\tau\right)\left(\lambda S\right)^{2-\beta}e^{(2-\beta)(r-q)\tau} \\
&= 2\frac{2\left(r-q\right)}{\left(2-\beta\right)\frac{\sigma^2}{\lambda^{\beta-2}S_t^{\beta-2}}\left[e^{(2-\beta)(r-q)\tau}-1\right]}\lambda^{2-\beta}S^{2-\beta}e^{(2-\beta)(r-q)\tau} \\
&= 2\frac{2\left(r-q\right)}{\left(2-\beta\right)\frac{\sigma^2}{S_t^{\beta-2}}\left[e^{(2-\beta)(r-q)\tau}-1\right]}S^{2-\beta}e^{(2-\beta)(r-q)\tau} \\
&= 2\kappa\left(\tau\right)S^{2-\beta}e^{(2-\beta)(r-q)\tau} \\
&= 2x\left(S_t;\tau\right). \qquad (8)
\end{aligned}$$

Finally, combining equation (4) arises after combining equations (5), (7) and (8).

(b) On the determination date, the value of the forward-start option is equal to the value of the corresponding standard option:

$$p_{T_1}^f\left(S_{T_1},X=\alpha S_{T_1},T_2\right)=p_{T_1}\left(S_{T_1},X=\alpha S_{T_1},T_2\right). \qquad (9)$$

On the other hand, equations (29) and (59) of the handouts imply that

$$\begin{aligned}
& p_{T_1}\left(S_{T_1},X=\alpha S_{T_1},T_2\right) \qquad (10) \\
=\; & -S_{T_1}e^{-q(T_2-T_1)}F_{\chi^2\left(2+\frac{2}{2-\beta},2x\left(S_{T_1};T_2-T_1\right)\right)}\left(2\kappa\left(T_2-T_1\right)\left(\alpha S_{T_1}\right)^{2-\beta}\right) \\
& +\alpha S_{T_1}e^{-r(T_2-T_1)}Q_{\chi^2\left(\frac{2}{2-\beta},2\kappa(T_2-T_1)\left(\alpha S_{T_1}\right)^{2-\beta}\right)}\left(2x\left(S_{T_1};T_2-T_1\right)\right),
\end{aligned}$$

where

$$\kappa\left(\tau\right):=\frac{2\left(r-q\right)}{\left(2-\beta\right)\delta^2\left[e^{(2-\beta)(r-q)\tau}-1\right]},$$

and

$$x\left(S;\tau\right):=\kappa\left(\tau\right)S^{2-\beta}e^{(2-\beta)(r-q)\tau}.$$

But since both probabilities on the right-hand side of equation (10) are homogeneous functions of degree zero on the spot and the strike, equation (10) can be restated as

$$\begin{aligned}
& p_{T_1}\left(S_{T_1},X=\alpha S_{T_1},T_2\right) \\
=\; & -S_{T_1}e^{-q(T_2-T_1)}F_{\chi^2\left(2+\frac{2}{2-\beta},2x(1;T_2-T_1)\right)}\left(2\kappa\left(T_2-T_1\right)\left(\alpha\right)^{2-\beta}\right) \\
& +\alpha S_{T_1}e^{-r(T_2-T_1)}Q_{\chi^2\left(\frac{2}{2-\beta},2\kappa(T_2-T_1)(\alpha)^{2-\beta}\right)}\left(2x\left(1;T_2-T_1\right)\right) \\
=\; & S_{T_1}\times p_{T_1}\left(1,X=\alpha,T_2\right). \qquad (11)
\end{aligned}$$

Combining equations (9) and (11), then

$$\begin{aligned} p_t^f\left(S_t, X=\alpha S_{T_1}, T_2\right) &= e^{-r(T_1-t)}\mathbb{E}_{\mathbb{Q}}\left[p_{T_1}^f\left(S_{T_1}, X=\alpha S_{T_1}, T_2\right)|\mathcal{F}_t\right] \\ &= e^{-r(T_1-t)}\mathbb{E}_{\mathbb{Q}}\left[S_{T_1}|\mathcal{F}_t\right]\times p_{T_1}\left(1, X=\alpha, T_2\right) \\ &= e^{-r(T_1-t)}S_t e^{(r-q)(T_1-t)}p_{T_1}\left(1, X=\alpha, T_2\right) \\ &= S_t e^{-q(T_1-t)}p_{T_1}\left(1, X=\alpha, T_2\right), \end{aligned}$$

i.e.

$$\begin{aligned} & p_t^f\left(S_t, X=\alpha S_{T_1}, T_2\right) \\ =\ & -S_t e^{-q(T_2-t)}F_{\chi^2\left(2+\frac{2}{2-\beta}, 2x(1;T_2-T_1)\right)}\left(2\kappa\left(T_2-T_1\right)(\alpha)^{2-\beta}\right) \\ & +\alpha S_t e^{-q(T_1-t)}e^{-r(T_2-T_1)}Q_{\chi^2\left(\frac{2}{2-\beta}, 2\kappa(T_2-T_1)(\alpha)^{2-\beta}\right)}\left(2x\left(1;T_2-T_1\right)\right). \end{aligned}$$

(c) Using the risk-neutral measure $\mathbb{Q}$ associated to the numeraire money-market account, the present value of the deposit is

$$\begin{aligned} V_t &= \mathbb{E}_{\mathbb{Q}}\left\{\exp\left(-\int_t^T r_s ds\right)\left[M+M\times E\left(T,T+\delta\right)\times\delta\right]|\mathcal{F}_t\right\} \\ &= MP\left(t,T\right)+M\mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s ds\right)\times E\left(T,T+\delta\right)\times\delta|\mathcal{F}_t\right]. \end{aligned} \tag{12}$$

Using the identity

$$P\left(T,T+\delta\right)=\left[1+\delta\times E\left(T,T+\delta\right)\right]^{-1},$$

equation (12) can be rewritten as

$$\begin{aligned} V_t &= MP\left(t,T\right)+M\mathbb{E}_{\mathbb{Q}}\left\{\exp\left(-\int_t^T r_s ds\right)\times\left[P\left(T,T+\delta\right)^{-1}-1\right]|\mathcal{F}_t\right\} \\ &= MP\left(t,T\right)+M\mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s ds\right)\times P\left(T,T+\delta\right)^{-1}|\mathcal{F}_t\right]-MP\left(t,T\right) \\ &= M\mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s ds\right)\times P\left(T,T+\delta\right)^{-1}|\mathcal{F}_t\right]. \end{aligned} \tag{13}$$

Changing to the forward measure $\mathbb{Q}_T$ associated to the numeraire $P\left(t,T\right)$, equation (13) becomes

$$\begin{aligned} V_t &= MP\left(t,T\right)\mathbb{E}_{\mathbb{Q}_T}\left[\frac{P\left(T,T+\delta\right)^{-1}}{P\left(T,T\right)}|\mathcal{F}_t\right] \\ &= MP\left(t,T\right)\frac{P\left(t,T+\delta\right)^{-1}}{P\left(t,T\right)} \\ &= MP\left(t,T+\delta\right)^{-1}. \end{aligned}$$

2. Via equação (59) dos handouts, o valor da put Europeia (com $\beta < 2$) é dado por:

$$p_t\left(S, X, T\right) = -S_t e^{-q(T-t)} F_{\chi^2\left(2+\frac{2}{2-\beta}, 2x\right)}\left(2\kappa X^{2-\beta}\right) + X e^{-r(T-t)} Q_{\chi^2\left(\frac{2}{2-\beta}, 2\kappa X^{2-\beta}\right)}\left(2x\right), \tag{14}$$

$$\kappa := \frac{2\left(r-q\right)}{\left(2-\beta\right)\delta^2\left[e^{(2-\beta)(r-q)(T-t)}-1\right]}, \tag{15}$$

e

$$x := \kappa S_t^{2-\beta} e^{(2-\beta)(r-q)(T-t)}. \tag{16}$$

Sendo a volatilidade da taxa de rentabilidade do activo subjacente é igual a 25% ao ano, então, e via equação (2) dos handouts,

$$\delta = \frac{25\%}{\left(10\right)^{\frac{1-2}{2}}} = 0.790569.$$

Retomando as equações (15) e (16),

$$\kappa = \frac{2\left(1\% - 2\%\right)}{\left(2-1\right)\left(0.790569\right)^2\left[e^{(2-1)(1\%-2\%)\times 2}-1\right]} \cong 1.616053333,$$

e

$$x = 1.616053333 \times \left(10\right)^{2-1} e^{(2-1)(1\%-2\%)\times 2} \cong 15.84053333.$$

(a) Substituindo na equação (14), então

$$\begin{aligned} p_t &= -10 \times e^{-2\%\times 2} \times F_{\chi^2\left(2+\frac{2}{2-1}, 2\times 15.84053333\right)}\left(2 \times 1.616053333 \times 10^{2-1}\right) \\ &\quad +10 \times e^{-1\%\times 2} \times Q_{\chi^2\left(\frac{2}{2-1}, 2\times 1.616053333\times 10^{2-1}\right)}\left(2 \times 15.84053333\right) \\ &= -10 \times e^{-2\%\times 2} \times F_{\chi^2(4,31.68106666)}\left(32.32106666\right) \\ &\quad +10 \times e^{-1\%\times 2} \times Q_{\chi^2(2,32.32106666)}\left(31.68106666\right). \end{aligned} \tag{17}$$

Via tabela do enunciado, sabemos que

$$F_{\chi^2(4,31.68106666)}\left(32.32106666\right) = 0.416953552. \tag{18}$$

A probabilidade $Q_{\chi^2(2,32.32106666)}\left(31.68106666\right)$ pode ser calculada via aproximação de Sankaran, i.e.

$$\begin{aligned} Q_{\chi^2(a,b)}\left(z\right) &= 1 - \mathbb{Q}\left(\chi^2\left(a,b\right) < z\right) \\ &= 1 - \mathbb{Q}\left\{\left[\frac{\chi^2\left(a,b\right)}{a+b}\right]^h < \left(\frac{z}{a+b}\right)^h\right\} \\ &\approx \Phi\left[-\frac{\left(\frac{z}{a+b}\right)^h - \mu_h}{\sigma_h}\right], \end{aligned} \tag{19}$$

onde

$$\mu_h := 1 + h\left(h-1\right)\frac{a+2b}{\left(a+b\right)^2} - h\left(h-1\right)\left(2-h\right)\left(1-3h\right)\frac{\left(a+2b\right)^2}{2\left(a+b\right)^4}, \tag{20}$$

$$\sigma_h^2 := h^2 \frac{2(a+2b)}{(a+b)^2} \left[1 - (1-h)(1-3h) \frac{a+2b}{(a+b)^2} \right], \tag{21}$$

e

$$h := 1 - \frac{2}{3}(a+b)(a+3b)(a+2b)^{-2}. \tag{22}$$

Visto que $a = 2$ e $b = 32.32106666$, então

$$\begin{aligned} h &= 1 - \frac{2}{3}(2 + 32.32106666)(2 + 3 \times 32.32106666) \\ &\quad (2 + 2 \times 32.32106666)^{-2} \\ &\cong 0.490146429, \end{aligned}$$

$$\begin{aligned} \mu_h &= 1 + 0.490146429 \times (0.490146429 - 1) \frac{2 + 2 \times 32.32106666}{(2 + 32.32106666)^2} \\ &\quad - 0.490146429 \times (0.490146429 - 1)(2 - 0.490146429) \\ &\quad (1 - 3 \times 0.490146429) \frac{(2 + 2 \times 32.32106666)^2}{2(2 + 32.32106666)^4} \\ &\cong 0.985577577, \end{aligned}$$

e

$$\begin{aligned} \sigma_h^2 &= 0.490146429^2 \times \frac{2(2 + 2 \times 32.32106666)}{(2 + 32.32106666)^2} \\ &\quad \left[1 - (1 - 0.490146429)(1 - 3 \times 0.490146429) \frac{2 + 2 \times 32.32106666}{(2 + 32.32106666)^2} \right] \\ &\cong 0.566897941. \end{aligned}$$

Utilizando a equação (19),

$$\Phi \left[- \frac{\left(\frac{z}{a+b} \right)^h - \mu_h}{\sigma_h} \right]$$

$$\begin{aligned} Q_{\chi^2(2,32.32106666)}(31.68106666) &= \Phi \left[- \frac{\left(\frac{31.68106666}{2+32.32106666} \right)^{0.490146429} - 0.985577577}{\sqrt{0.566897941}} \right] \\ &\cong 0.557599022. \end{aligned} \tag{23}$$

Finalmente, combinando as equações (17), (18) e (23),

$$\begin{aligned} p_t &= -10 \times e^{-2\% \times 2} \times 0.416953552 + 10 \times e^{-1\% \times 2} \times 0.557599022 \\ &\cong EUR1.45953. \end{aligned}$$

(a) Utilizando a Proposição 22 dos apontamentos,

$$\begin{aligned} p_0 &= -10 \times e^{-2\% \times 1} \times \left[1 - P_1\left(S_t = 10, v_t = 0.09; T = 1, X = 10\right)\right] \\ &\quad + e^{-1\% \times 1} \times 10 \times \left[1 - P_2\left(S_t = 10, v_t = 0.09; T = 1, X = 10\right)\right]. \end{aligned} \tag{24}$$

Com base nas equações (173) e (174) dos apontamentos:

$$\begin{aligned} P_1\left(S_t = 10, v_t = 0.09; T = 1, X = 10\right) &\approx \frac{1}{2} + \frac{0.11419742}{\pi} \\ &\cong 5.3635E - 01, \end{aligned} \tag{25}$$

e

$$\begin{aligned} P_2\left(S_t = 10, v_t = 0.09; T = 1, X = 10\right) &\approx \frac{1}{2} + \frac{-0.17224869}{\pi} \\ &\cong 4.4517E - 01. \end{aligned} \tag{26}$$

Combinando as equações (24), (25) e (26),

$$\begin{aligned} p_0 &= -10 \times e^{-2\% \times 1} \times (1 - 5.3635E - 01) + 10 \times e^{-1\% \times 2} \times (1 - 4.4517E - 01) \\ &\cong EUR0.94839. \end{aligned}$$

(b) O payoff terminal de uma range asset-or-nothing é dado por

$$RAN_T\left(S, X, T\right) = S_T \mathbb{1}_{\{X_a < S_T < X_b\}}.$$

Consequentemente,

$$\begin{aligned} RAN_t\left(S, X, T\right) &= e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}\left(S_T \mathbb{1}_{\{X_a < S_T < X_b\}} \middle| \mathcal{F}_t\right) \\ &= S_t e^{qt} \mathbb{E}_{\mathbb{Q}_S}\left(\left.\frac{S_T \mathbb{1}_{\{X_a < S_T < X_b\}}}{S_T e^{qT}}\right| \mathcal{F}_t\right) \\ &= S_t e^{-q(T-t)} \mathbb{Q}_S\left(X_a < S_T < X_b \middle| \mathcal{F}_t\right) \\ &= S_t e^{-q(T-t)} \left[\mathbb{Q}_S\left(S_T < X_b \middle| \mathcal{F}_t\right) - \mathbb{Q}_S\left(S_T < X_a \middle| \mathcal{F}_t\right)\right]. \end{aligned} \tag{27}$$

Visto que

$$\mathbb{Q}_S\left(S_T < X \middle| \mathcal{F}_t\right) = 1 - P_1\left(S_t, v_t; T, X\right), \tag{28}$$

então combinando as equações (27) e (28),

$$RAN_t\left(S, X, T\right) = S_t e^{-q(T-t)} \left[P_1\left(S_t, v_t; T, X_a\right) - P_1\left(S_t, v_t; T, X_b\right)\right]. \tag{29}$$

No caso em apreço

$$P_1\left(S_t, v_t; T, X_a = 10\right) \cong 5.3635E - 01$$

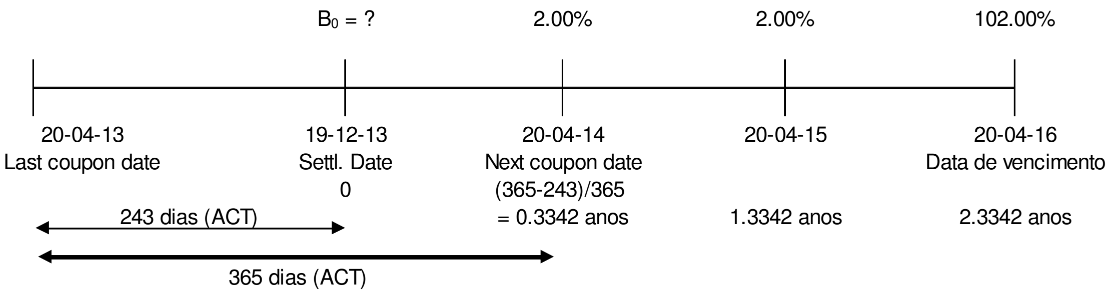
e

$$\begin{aligned} P_1\left(S_t, v_t; T, X_b = 12\right) &\approx \frac{1}{2} + \frac{-0.83281298}{\pi} \\ &\cong 2.3491E - 01. \end{aligned}$$

Portanto,

$$\begin{aligned} RAN_0 &= 10 \times e^{-2\% \times 2} \times (5.3635E - 01 - 2.3491E - 01) \\ &\cong EUR2.955. \end{aligned}$$

(a) Pretende-se avaliar uma obrigação com os seguintes cash flows vincendos:



Portanto,

$$\begin{aligned} B_0 &= 2\%P(0,0.3342)+2\%P(0,1.3342)+102\%P(0,2.3342) \\ &= 2\%P(0,0.3342)+2\%\times 0.9769+102\%\times 0.9577. \qquad (30) \end{aligned}$$

Relativamente ao factor de desconto a 0.3342 anos, via equações (269) e (270) dos apontamentos,

$$\begin{aligned} B(0.3342) &= \frac{1-e^{-3\times 0.3342}}{3} \\ &\cong 0.2110, \end{aligned}$$

e

$$\begin{aligned} A(0.3342) &= (0.211-0.3342)\left(2\%-\frac{0.03^2}{2(3)^2}\right)-\frac{0.03^2}{0.2795\times 3}\times(0.211)^2 \\ &\cong -0.0025. \end{aligned}$$

Portanto,

$$\begin{aligned} P(0,0.3342) &= \exp(-0.0025-0.211\times 1\%) \\ &\cong 0.9954. \end{aligned}$$

Retomando a equação (30), então

$$\begin{aligned} B_0 &= 2\%\times 0.9954+2\%\times 0.9769+102\%\times 0.9577 \\ &\cong 101.63\%. \end{aligned}$$

(b) Utilizando a Proposição 59 dos apontamentos,

$$\begin{aligned} & c_0\left[P(0,2.3342);96.12\%;0.3342\right] \\ =\ & P(0,2.3342)\,\Phi\left(d_1^V\right)-96.12\%P(0,0.3342)\,\Phi\left(d_0^V\right) \\ =\ & 0.9577\times\Phi\left(d_1^V\right)-0.9612\times 0.9954\times\Phi\left(d_0^V\right), \end{aligned}$$

onde

$$\begin{aligned} v\left(0, 0.3342, 2.3342\right) &= \sqrt{\frac{0.03^2}{3^2}\left[1-e^{-3\times 2}\right]^2 \frac{1-e^{-2\times 3\times 0.3342}}{2\times 3}} \\ &\cong 0.379\%, \end{aligned}$$

$$\begin{aligned} d_1^V &= \frac{\ln\left(\frac{0.9577}{0.9612\times 0.9954}\right)+\frac{(0.379\%)^2}{2}}{0.379\%} \\ &\cong 0.243954083, \end{aligned}$$

e

$$\begin{aligned} d_0^V &= 0.243954083 - 0.379\% \\ &\cong 0.240165681. \end{aligned}$$

Portanto,

$$\begin{aligned} & c_0\left[P\left(0, 2.3342\right); 96.12\%; 0.3342\right] \\ = \ & 0.9577 \times \Phi\left(0.243954083\right) - 0.9612 \times 0.9954 \times \Phi\left(0.240165681\right) \\ = \ & 0.9577 \times 0.596366809 - 0.9612 \times 0.9954 \times 0.594899091 \\ \cong \ & 0.193\%. \end{aligned} \tag{31}$$

(c) De acordo com a Proposição 61 dos apontamentos, o valor actual de uma *call* sobre a CBB pode ser decomposto numa carteira de 2 *calls* Europeias sobre PBD:

$$\begin{aligned} & c_0\left(B_t; X = 100\%; T = 0.3342\right) \\ = \ & 2\% \times c_0\left[P\left(0, 1.3342\right); X_1; T = 0.3342\right] \\ & +102\% \times c_0\left[P\left(0, 2.3342\right); X_2; T = 0.3342\right]. \end{aligned} \tag{32}$$

Os *strikes* podem ser obtidos via equação (327) dos apontamentos:

$$\begin{aligned} X_1 &= \exp\left[A\left(1.3342 - 0.3342\right) - B\left(1\right) \times 1.905\%\right] \\ &= \exp\left(-0.0136 - 0.3167 \times 1.905\%\right) \\ &\cong 98.05\%, \end{aligned}$$

e

$$\begin{aligned} X_2 &= \exp\left[A\left(2.3342 - 0.3342\right) - B\left(2\right) \times 1.905\%\right] \\ &= \exp\left(-0.0333 - 0.3325 \times 1.905\%\right) \\ &\cong 96.12\%. \end{aligned}$$

Portanto,

$$\begin{aligned} & c_0\left(B_t; X = 100\%; T = 0.3342\right) \\ = \ & 2\% \times c_0\left[P\left(0, 1.3342\right); 98.05\%; T = 0.3342\right] \\ & +102\% \times c_0\left[P\left(0, 2.3342\right); 96.12\%; T = 0.3342\right]. \end{aligned} \tag{33}$$

A segunda *call* já foi avaliada na alínea anterior –vide equação (31):

$$c_0\left[P\left(0,2.3342\right);96.12\%;T=0.3342\right]\cong 0.193\%. \tag{34}$$

Relativamente à primeira *call*, o enunciado fornece o seguinte valor actual:

$$c_0\left[P\left(0,1.3342\right);98.05\%;T=0.3342\right]=0.187\%. \tag{35}$$

Combinando as equações (33), (34) e (35),

$$\begin{aligned}
& c_0\left(B_t;X=100\%;T=0.3342\right)\\
= \ & 2\%\times 0.187\%+102\%\times 0.193\%\\
\cong \ & 0.2\%.
\end{aligned}$$

(a) Seja $t=0$, $T_1=1$ ano, $K=\frac{P(0,2)}{P(0,1)}=\frac{0.9632}{0.9826}\cong 0.9802$, $M=\$1M$ e $P\left(T_1,T_2\right)$ o valor daqui a 1 ano de uma obrigação de cupão zero com vencimento daqui a 2 anos (T_2). Consequentemente, o payoff terminal da opção, daqui a 1 ano, é dado por

$$V_{T_1}=M\times P\left(T_1,T_2\right)\times \mathbb{1}_{\{P(T_1,T_2)>K\}}.$$

Consequentemente, o valor actual da opção é dado por

$$\begin{aligned}
V_t &= P\left(t,T_2\right)\times \mathbb{E}_{\mathbb{Q}_{T_2}}\left[\left.\frac{M\times P\left(T_1,T_2\right)\times \mathbb{1}_{\{P(T_1,T_2)>K\}}}{P\left(T_1,T_2\right)}\right|\mathcal{F}_t\right]\\
&= M\times P\left(t,T_2\right)\times \mathbb{E}_{\mathbb{Q}_{T_2}}\left[\left.\mathbb{1}_{\{P(T_1,T_2)>K\}}\right|\mathcal{F}_t\right]\\
&= M\times P\left(t,T_2\right)\times \mathbb{Q}_{T_2}\left[\left.P\left(T_1,T_2\right)>K\right|\mathcal{F}_t\right].
\end{aligned}$$

Visto que

$$P\left(T_1,T_2\right)=\exp\left[A\left(T_2-T_1\right)+B\left(T_2-T_1\right)r_{T_1}\right],$$

então

$$\begin{aligned}
V_t &= M\times P\left(t,T_2\right)\times \mathbb{Q}_{T_2}\left[\left.A\left(T_2-T_1\right)+B\left(T_2-T_1\right)r_{T_1}>\ln\left(K\right)\right|\mathcal{F}_t\right]\\
&= M\times P\left(t,T_2\right)\times \mathbb{Q}_{T_2}\left[\left.r_{T_1}<\frac{\ln\left(K\right)-A\left(T_2-T_1\right)}{B\left(T_2-T_1\right)}\right|\mathcal{F}_t\right]\\
&= M\times P\left(t,T_2\right)\times F_{\chi^2\left(\frac{4k\theta}{\sigma^2},\zeta\right)}\left[\frac{\ln\left(K\right)-A\left(T_2-T_1\right)}{B\left(T_2-T_1\right)L}\right],
\end{aligned} \tag{36}$$

visto que

$$\frac{r_{T_1}}{L}\overset{\mathbb{Q}_{T_2}}{\sim}\chi^2\left(\frac{4k\theta}{\sigma^2},\zeta\right),$$

com

$$\gamma=\sqrt{\left(4\right)^2+2\times\left(0.1\right)^2}\cong 4.0025,$$

$$
\begin{aligned}
\zeta &= \frac{8 r_t \gamma^2 e^{\gamma(T_1-t)}}{\sigma^2 \left[e^{\gamma(T_1-t)}-1\right]\left\{\gamma\left[e^{\gamma(T_1-t)}+1\right]+\left[k-\sigma^2\times B\left(T_2-T_1\right)\right]\left[e^{\gamma(T_1-t)}-1\right]\right\}} \\
&= \frac{8\times 1\%\times(4.0025)^2\times e^{4.0025\times 1}}{(0.1)^2\left(e^{4.0025\times 1}-1\right)\{4.0025\times\left[e^{4.0025\times 1}+1\right]+\left[4-(0.1)^2\times(-0.2454)\right]\left[e^{4.0025\times 1}-1\right]} \\
&\cong 0.297946479
\end{aligned}
$$

e

$$
\begin{aligned}
L &= \frac{\sigma^2}{2}\frac{e^{\gamma(T_1-t)}-1}{\gamma\left[e^{\gamma(T_1-t)}+1\right]+\left[k-\sigma^2\times B\left(T_2-T_1\right)\right]\left[e^{\gamma(T_1-t)}-1\right]} \\
&= \frac{(0.1)^2}{2}\frac{e^{4.0025\times 1}-1}{4.0025\times\left[e^{4.0025\times 1}+1\right]+\left[4-(0.1)^2\times(-0.2454)\right]\left[e^{4.0025\times 1}-1\right]} \\
&\cong 0.000613202.
\end{aligned}
$$

Portanto,

$$
\begin{aligned}
& F_{\chi^2\left(\frac{4k\theta}{\sigma^2},\zeta\right)}\left[\frac{\ln(K)-A\left(T_2-T_1\right)}{B\left(T_2-T_1\right)L}\right] \\
= \; & F_{\chi^2\left(\frac{4\times 4\times 2\%}{0.1^2},0.297946479\right)}\left[\frac{\ln(0.9802)-(-0.0151)}{-0.2454\times 0.000613202}\right] \\
= \; & F_{\chi^2(32,0.297946479)}(32.30285158) \\
\cong \; & 0.361963928.
\end{aligned}
$$

Finalmente, recuperando a equação (36), então

$$
\begin{aligned}
V_t &= \$1M\times 0.9632\times 0.533444162 \\
&\cong EUR513,814.83.
\end{aligned}
$$

(b) The new model is a multi-factor CIR with two factors ($n=2$). Therefore,

$$
r_t = f + G'\cdot Y_t,
$$

with $f=0$, $G=\left[\begin{array}{cc}1 & 1\end{array}\right]'$, $Y_t=\left[\begin{array}{cc}1\% & 0.25\%\end{array}\right]'$, and

$$
\begin{aligned}
dY_t &= \left(\left[\begin{array}{c}4\times 2\% \\ 8\%\end{array}\right]+\left[\begin{array}{cc}-4 & 0 \\ 0 & -4\end{array}\right]\cdot Y_t\right)dt \\
&\quad +\left[\begin{array}{cc}10\% & 0 \\ 0 & 10\%\end{array}\right]\cdot\left[\begin{array}{cc}\sqrt{Y_{1,t}} & 0 \\ 0 & \sqrt{Y_{2,t}}\end{array}\right]\cdot\left[\begin{array}{c}dW^{\mathbb{Q}}_{1,t} \\ dW^{\mathbb{Q}}_{1,t}\end{array}\right].
\end{aligned}
$$

(c) Under a multi-factor CIR, the discount factor is equal to the product of the discount factors associated to each square-root process. Therefore,

$$
\begin{aligned}
P(0,2) &= \exp\left[A(2)+B(2)\times 1\%\right]\times\exp\left[A(2)+B(2)\times 0.25\%\right] \\
&= \exp\left[2\times(-0.0350)+(-0.2498)\times(1\%+0.25\%)\right] \\
&\cong 0.929499381.
\end{aligned}
$$