Modelos de Estrutura Temporal de Taxas de Juro
Mestrado em Matemática Financeira 17/18
IBS e FCUL

Exame 1ª Época

18/Dez/18 Duração: 3h

Case 1 Please answer only two of the following questions: (2x2V)

a) Under the Vasiček (1977) model, compute the (time t) *fair value* of an asset-or-nothing put with expiry date at time T_1 ($\geq t$), strike K, and on a zero-coupon bond $P(t, T_2)$ with maturity at time T_2 ($\geq T_1$).

b) Under the Heston (1993) model, find the time-t fair value of a European forward-start put on the rate of return of an asset with a spot price S_t, with a strike price βS_{T_1} and with expiry date on time T_2 ($\geq T_1 \geq t$).

c) Under a multi-factor CIR model, compute the moment generating function of the instantaneous interest rate integral under the risk-neutral measure $\mathbb{Q}$, i.e. the expectation $\mathbb{E}_{\mathbb{Q}}\left[\exp\left(\mu \int_t^T r(u)\, du\right) \middle| \mathcal{F}_t\right]$, for $\mu \in \mathbb{R}$. For this purpose, consider that, under the multi-factor CIR model

$$r(t) = \sum_{j=1}^{n} Y_j(t),$$

and

$$dY_j(t) = k_j(\theta_j - Y_j(t))\, dt + \sigma_j \sqrt{Y_j(t)} dW_j^{\mathbb{Q}}(t),$$

for $j = 1, \ldots, n$, while all Brownian motions $\left\{W_j^{\mathbb{Q}}(t), t \geq 0\right\}$ are independent.

Case 2 Consider a CEV process given by the following SDE:

$$dS_t = (r - q) S_t dt + \delta S_t^{\frac{\beta}{2}} dW_t^{\mathbb{Q}}.$$

Assume that $S = \$100$, $\beta = -2$, $r = 1\%$, $q = 3\%$ and that the (annualized) standard deviation of stock returns is equal to 30%. Please consider also the following table containing cumulative probabilities associated to a noncentral chi-square random variable with 0.5 degrees of freedom and a noncentrality parameter equal to 4.616138739:

x	0.50000	4.61614	5.44519
F(x)	0.09460	0.55141	0.62220

Please answer the following questions:

a) Price a European-style standard call on the stock S, with strike equal to \$95 and with a time-to-maturity of 0.5 years. (2V)

b) Price an ATM European-style asset-or-nothing call on the stock S, with a time-to-maturity of 0.5 years, and with a contract size of 10 shares. (1V)

Case 3 Consider the following parameters for the Heston (1993) model:

- Spot price of the ESC stock = EUR100;
- *Dividend yield* for the ESC stock (continuous compounding) = 3% (30/360);
- Risk-free interest rate (continuous compounding) = 1% (30/360);
- Instantaneous variance of the stock returns (v) = 0.06;
- Speed of mean reversion of the volatility (k) = 2;
- Long-term level of the instantaneous variance (θ) = 0.04;
- Volatility of the instantaneous variance (σ) = 10%; and
- Correlation coefficient between the stock price and the instantaneous variance (ρ) = -0.4.

Next table summarizes the implementation of equations (173) and (174) of the handouts for the strikes EUR95 and EUR100, for a maturity of 6 months, and through a Gauss-Laguerre quadrature with 15 nodes:

w_i	ϕ_i	$X = 95$		$X = 100$	
		$f_1(\phi_i)$	$f_2(\phi_i)$	$f_1(\phi_i)$	$f_2(\phi_i)$
2.1823E-01	9.3308E-02	5.9664E-02	3.0881E-02	3.3611E-03	-2.5422E-02
3.4221E-01	4.9269E-01	8.8711E-02	4.5937E-02	5.0323E-03	-3.7744E-02
2.6303E-01	1.2156E+00	1.8016E-01	9.3518E-02	1.0594E-02	-7.6074E-02
1.2643E-01	2.2699E+00	4.9529E-01	2.5895E-01	3.2162E-02	-2.0447E-01
4.0207E-02	3.6676E+00	1.8179E+00	9.6556E-01	1.4281E-01	-7.1282E-01
8.5639E-03	5.4253E+00	8.7333E+00	4.7741E+00	9.0817E-01	-3.0947E+00
1.2124E-03	7.5659E+00	5.3410E+01	3.0569E+01	7.8051E+00	-1.5725E+01
1.1167E-04	1.0120E+01	3.9973E+02	2.4449E+02	8.4210E+01	-8.3638E+01
6.4599E-06	1.3130E+01	3.4539E+03	2.3129E+03	1.0594E+03	-3.3339E+02
2.2263E-07	1.6654E+01	3.1363E+04	2.3827E+04	1.4413E+04	1.7410E+03
4.2274E-09	2.0776E+01	2.4834E+05	2.3248E+05	1.9310E+05	7.9961E+04
3.9219E-11	2.5624E+01	8.2697E+05	1.5316E+06	2.1635E+06	1.4454E+06
1.4565E-13	3.1408E+01	-1.4107E+07	-3.6952E+06	1.1968E+07	1.4564E+07
1.4830E-16	3.8531E+01	-4.6981E+07	-1.0047E+08	-1.1322E+08	-1.6043E+07
1.6006E-20	4.8026E+01	4.9514E+08	9.2878E+08	4.3006E+08	-4.5474E+08
	$\sum_{i=1}^{15} w_i f_j(\phi_i) =$	0.44103501	0.24515768	0.05265042	-0.14926188

Please answer the following questions:

a) Price a European-style put on the ESC stock, with a strike price equal to EUR95, and with a time-to-maturity of 0.5 years. (1V)

b) Price a volatility derivative that pays, after 6 months, the exponential of the instantaneous variance of the ESC stock return observed at that time. (1V)

Case 4 Consider the following parameters, estimated under measure $\mathbb{Q}$, for the Vasiček (1977) model, and using German treasury bonds for the settlement date of 18/12/18:

alpha	4
gamma	2%
rho	10%
r(t)	1.0%

Next table shows discount factors (for different maturities; ACT/ACT) based on the previous model parameters:

T-t	B(t,T)	A(t,T)	P(t,T)
0.5	0.2162	-0.0056	0.9923
1	0.2454	-0.0149	0.9828
1.4027	0.2491	-0.0228	0.9751
2	0.2499	-0.0345	0.9637
2.4027	0.2500	-0.0424	0.9561
3	0.2500	-0.0542	0.9449

Please answer the following questions:

a) Find the fair value of a treasury bond with maturity at 14/05/2021, with *bullet* redemption, and with a coupon rate of 0.25% (annual coupon under the daycount convention ACT/ACT). For this purpose, consider that the number of calendar days between 14/05/2018 and 18/12/2018 is equal to 218 days. (2V)

b) Formulate a trading decision, knowing that the treasury bond possesses a clean price equal to 96.10%(bid)-96.15%(ask). (1V)

c) Price a European-style put, with strike equal to 96.513%, maturity at 14/05/2019 and on a Treasury Bill with maturity at 14/05/2021. (2V)

d) Price a European-style put with a maturity at 14/05/2019, with a strike equal to 97%, and on the coupon-bearing bond described in question a). For this purpose, consider that an instantaneous interest rate of 0.401% yields, on 14/05/2019, a fair value of 97% for the underlying coupon-bearing bond, and that the market trades European-style options with a time-to-maturity of 0.4027 years, and on German treasury zero-coupon bonds with a time-to-maturity of 1.4027 years: (3V)

strike	96.513%	98.425%
Call	1.582%	0.193%
Put	0.009%	0.520%

Case 5 Consider the following parameters for the Cox, Ingersoll and Ross (1985) model, estimated under measure $\mathbb{Q}$ and using IRS quotes:

k	4.0
theta	2.0%
sigma	10.0%
r	1.0%

Next table contains discount factors (for different maturities) based on the previous model' parameters:

T-t	B(T-t)	A(T-t)	P(t,T)
0.5	-0.21613179	-0.00567617	0.99219314
1	?	-0.01508895	?
1.5	-0.24930456	-0.02500687	0.97287476
2	-0.24983848	-0.03499308	0.96320264
2.5	-0.24991064	-0.04498851	0.95362228
3	-0.24992040	-0.05498519	0.94413662

Consider also the following table containing cumulative probabilities associated to a noncentral chi-square random variable with 32 degrees of freedom and a noncentrality parameter equal to b:

	F(x)		
x	2.5029043	34.4979203	34.5072145
b			
2.50223017	2.4857E-12	0.53277392	0.53320341
2.50290430	2.4852E-12	0.53274266	0.53317214

Please answer the following questions:

a) Find a quote for a FRA 6x12 under the 30/360 daycount convention. (1V)

b) Price an ATM-forward European-style put with a time-to-maturity of 0.5 years, and on a zero-coupon bond with a time-to-maturity of two years, knowing that $L_1 = 0.000540329$ and $\zeta_1 = 2.502904304$. (2V)

References

Cox, J., J. Ingersoll, and S. Ross, 1985, A Theory of the Term Structure of Interest Rates, *Econometrica* 53, 385–407.

Heston, S., 1993, A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, *Review of Financial Studies* 6, 327–343.

Vasiček, O., 1977, An Equilibrium Characterization of the Term Structure, *Journal of Financial Economics* 5, 177–188.