Modelos de Estrutura Temporal de Taxas de Juro
Mestrado em Matemática Financeira 07/08
IBS e FCUL
Exame 1ª Época - Resolução

12/Dez/08 Duração: 3h

1. (a) A ODE

$$\frac{d\psi\left(t-t_0\right)}{-\frac{1}{2}\sigma^2\psi^2\left(t-t_0\right)-k\psi\left(t-t_0\right)+\mu}=dt,$$

pode ser reescrita como

$$\frac{d\psi\left(t-t_0\right)}{-\frac{\sigma^2}{2}\left[\psi\left(t-t_0\right)+\frac{k+h}{\sigma^2}\right]\left[\psi\left(t-t_0\right)+\frac{k-h}{\sigma^2}\right]}=dt, \tag{1}$$

visto que $-\frac{k\pm h}{\sigma^2}$ são as duas raízes da seguinte equação do 2º grau:

$$-\frac{1}{2}\sigma^2\psi^2\left(t-t_0\right)-k\psi\left(t-t_0\right)+\mu=0.$$

Uma vez que

$$\begin{aligned}\frac{1}{-\frac{\sigma^2}{2}\left[\psi\left(t-t_0\right)+\frac{k+h}{\sigma^2}\right]\left[\psi\left(t-t_0\right)+\frac{k-h}{\sigma^2}\right]} &= \frac{a}{\psi\left(t-t_0\right)+\frac{k+h}{\sigma^2}}+\frac{b}{\psi\left(t-t_0\right)+\frac{k-h}{\sigma^2}}\\ &\Longrightarrow \left\{\begin{array}{c}\frac{k-h}{\sigma^2}a+\frac{k+h}{\sigma^2}b=-\frac{2}{\sigma^2}\\ a+b=0\end{array}\right.\\ &\Longleftrightarrow \left\{\begin{array}{c}b=-\frac{1}{h}\\ a=\frac{1}{h}\end{array}\right.,\end{aligned}$$

então a equação (1) pode ser reescrita como

$$\frac{1}{h}\frac{d\psi\left(t-t_0\right)}{\psi\left(t-t_0\right)+\frac{k+h}{\sigma^2}}-\frac{1}{h}\frac{d\psi\left(t-t_0\right)}{\psi\left(t-t_0\right)+\frac{k-h}{\sigma^2}}=dt.$$

Portanto,

$$\frac{1}{h}\ln\left|\psi\left(t-t_0\right)+\frac{k+h}{\sigma^2}\right|-\frac{1}{h}\ln\left|\psi\left(t-t_0\right)+\frac{k-h}{\sigma^2}\right|=t+C, \tag{2}$$

sendo C uma constante de integração. Resolvendo a equação (2) em relação a $\psi\left(t-t_0\right)$:

$$\frac{1}{h}\ln\left|\frac{\psi\left(t-t_0\right)+\frac{k+h}{\sigma^2}}{\psi\left(t-t_0\right)+\frac{k-h}{\sigma^2}}\right|=t+C, \tag{3}$$

i.e.

$$\psi\left(t-t_0\right)+\frac{k+h}{\sigma^2}=\psi\left(t-t_0\right)e^{h(t+C)}+\frac{k-h}{\sigma^2}e^{h(t+C)},$$

ou

$$\psi\left(t-t_0\right)+\frac{k+h}{\sigma^2}=-\psi\left(t-t_0\right)e^{h(t+C)}-\frac{k-h}{\sigma^2}e^{h(t+C)}.$$

Consequentemente,

$$\begin{aligned}
\psi\left(t-t_0\right) &= \frac{\frac{k-h}{\sigma^2}e^{h(t+C)}-\frac{k+h}{\sigma^2}}{1-e^{h(t+C)}} \\
&= \frac{(k-h)\,e^{h(t+C)}-(k+h)}{\sigma^2\left[1-e^{h(t+C)}\right]},
\end{aligned} \tag{4}$$

ou

$$\begin{aligned}
\psi\left(t-t_0\right) &= \frac{-\frac{k-h}{\sigma^2}e^{h(t+C)}-\frac{k+h}{\sigma^2}}{1+e^{h(t+C)}} \\
&= \frac{-(k-h)\,e^{h(t+C)}-(k+h)}{\sigma^2\left[1+e^{h(t+C)}\right]}.
\end{aligned} \tag{5}$$

Impondo a condição terminal $\psi(0)=\lambda$ às equações (4) e (5), então

$$\lambda\sigma^2\left[1-e^{ht_0}e^{hC}\right]=(k-h)\,e^{ht_0}e^{hC}-(k+h)\,,$$

i.e.

$$e^{hC}=e^{-ht_0}\frac{\lambda\sigma^2+k+h}{\lambda\sigma^2+k-h}, \tag{6}$$

ou

$$\lambda\sigma^2\left[1+e^{ht_0}e^{hC}\right]=-(k-h)\,e^{ht_0}e^{hC}-(k+h)\,,$$

i.e.

$$e^{hC}=-e^{-ht_0}\frac{\lambda\sigma^2+k+h}{\lambda\sigma^2+k-h}. \tag{7}$$

Combinando as equações (4) e (6),

$$\begin{aligned}
\psi\left(t-t_0\right) &= \frac{(k-h)\,e^{ht}e^{-ht_0}\frac{\lambda\sigma^2+k+h}{\lambda\sigma^2+k-h}-(k+h)}{\sigma^2\left[1-e^{ht}e^{-ht_0}\frac{\lambda\sigma^2+k+h}{\lambda\sigma^2+k-h}\right]} \\
&= \frac{(k-h)\,e^{h(t-t_0)}\left(\lambda\sigma^2+k+h\right)-(k+h)\left(\lambda\sigma^2+k-h\right)}{\sigma^2\left(\lambda\sigma^2+k-h\right)-\sigma^2e^{h(t-t_0)}\left(\lambda\sigma^2+k+h\right)} \\
&= \frac{e^{h(t-t_0)}\left[\lambda\sigma^2(k-h)-2\sigma^2\mu\right]-\left[\lambda\sigma^2(k+h)-2\sigma^2\mu\right]}{\sigma^2\lambda\sigma^2\left[1-e^{h(t-t_0)}\right]+\sigma^2\left[k-h-(k+h)\,e^{h(t-t_0)}\right]} \\
&= \frac{e^{h(t-t_0)}\left[\lambda(k-h)-2\mu\right]-\lambda(k+h)+2\mu}{\lambda\sigma^2\left[1-e^{h(t-t_0)}\right]+k-h-(k+h)\,e^{h(t-t_0)}} \\
&= \frac{\lambda\left[h+k+(h-k)\,e^{h(t-t_0)}\right]+2\mu\left[e^{h(t-t_0)}-1\right]}{\sigma^2\lambda\left[e^{h(t-t_0)}-1\right]+h-k+(k+h)\,e^{h(t-t_0)}}.
\end{aligned}$$

(b) O payoff final de um floorlet com vencimento no momento t_{i+1}, sobre a taxa de juro nominal em vigor entre os momentos t_i ($> t_{i+1}$) e t_{i+1}, com uma flor rate igual a k e com um contract size unitário é dado por

$$Floorlet\,(t_{i+1}) = (t_{i+1} - t_i) \times [k - E\,(t_i, t_{i+1})]^+ .$$

Consequentemente,

$$\begin{aligned} Floorlet\,(t_i) &= P\,(t_i, t_{i+1})\,(t_{i+1} - t_i)\,\mathbb{E}_{\mathbb{Q}_{i+1}} \left\{ \left. \frac{[k - E\,(t_i, t_{i+1})]^+}{P\,(t_{i+1}, t_{i+1})} \right| \mathcal{F}_{t_i} \right\} \\ &= P\,(t_i, t_{i+1})\,\mathbb{E}_{\mathbb{Q}_{i+1}} \left\{ \left. [(t_{i+1} - t_i)\,k - (t_{i+1} - t_i)\,E\,(t_i, t_{i+1})]^+ \right| \mathcal{F}_{t_i} \right\} \end{aligned} \tag{8}$$

Por outro lado,

$$P\,(t_i, t_{i+1}) = \frac{1}{1 + (t_{i+1} - t_i)\,E\,(t_i, t_{i+1})}. \tag{9}$$

Combinando as equações (8) e (9),

$$\begin{aligned} Floorlet\,(t_i) &= P\,(t_i, t_{i+1})\,\mathbb{E}_{\mathbb{Q}_{i+1}} \left\{ \left. \left[(t_{i+1} - t_i)\,k - \frac{1}{P\,(t_i, t_{i+1})} + 1 \right]^+ \right| \mathcal{F}_{t_i} \right\} \\ &= [1 + (t_{i+1} - t_i)\,k]\,\mathbb{E}_{\mathbb{Q}_{i+1}} \left\{ \left. \left[P\,(t_i, t_{i+1}) - \frac{1}{1 + (t_{i+1} - t_i)\,k} \right]^+ \right| \mathcal{F}_{t_i} \right\} \\ &= [1 + (t_{i+1} - t_i)\,k]\,c_{t_i} \left[P\,(t_i, t_{i+1})\,; \frac{1}{1 + (t_{i+1} - t_i)\,k}; t_i \right]. \end{aligned}$$

Portanto,

$$Floorlet\,(0) = [1 + (t_{i+1} - t_i)\,k]\,c_0 \left[P\,(0, t_{i+1})\,; \frac{1}{1 + (t_{i+1} - t_i)\,k}; t_i \right].$$

(c) O payoff a liquidar daqui a 2 anos é igual a

$$EUR1,000,000 \times E_{6M}\,(1.5) = EUR2,000,000 \times \frac{E_{6M}\,(1.5)}{2},$$

onde $E_{6M}\,(1.5)$ designa a Euribor a 6 meses em vigor daqui a 1.5 anos. Portanto, o valor actual de tal payoff é dado por

$$EUR2,000,000 \times [P\,(0, 1.5) - P\,(0, 2)]\,,$$

onde $P\,(0, t)$ representa o factor de desconto interbancário a "t" anos. Operações financeiras a efectuar: i) Contrair financiamento a 2 anos e no valor de EUR2M; ii) Efectuar uma aplicação financeira a 1.5 anos e no valor de EUR2M; reinvestir, daqui a 1.5 anos, o valor acumulado desta aplicação por mais 6 meses.

(a) O payoff terminal de uma opção range asset-or-nothing é dado por

$$RA_T\left(S,X,T\right)=MS_T\mathbb{1}_{\{X_a<S_T<X_b\}}.$$

Consequentemente,

$$\begin{aligned}RA_t\left(S,X,T\right)&=e^{-r(T-t)}M\mathbb{E}_{\mathbb{Q}}\left(S_T\mathbb{1}_{\{X_a<S_T<X_b\}}\middle|\mathcal{F}_t\right)\\&=S_te^{qt}M\mathbb{E}_{\mathbb{Q}_S}\left(\frac{S_T\mathbb{1}_{\{X_a<S_T<X_b\}}}{S_Te^{qT}}\middle|\mathcal{F}_t\right)\\&=MS_te^{-q(T-t)}\mathbb{Q}_S\left(X_a<S_T<X_b\middle|\mathcal{F}_t\right)\\&=MS_te^{-q(T-t)}\left[\mathbb{Q}_S\left(S_T<X_b\middle|\mathcal{F}_t\right)-\mathbb{Q}_S\left(S_T<X_a\middle|\mathcal{F}_t\right)\right].\end{aligned}\tag{10}$$

Combinando as equações 58 e 106 dos handouts,

$$\mathbb{Q}_S\left(S_T<X\middle|\mathcal{F}_t\right)=1-Q_{\chi^2\left(2+\frac{2}{2-\beta},2x\right)}\left(2\kappa X^{2-\beta}\right),\tag{11}$$

para $\beta<2$ e sendo

$$\kappa:=\frac{2\left(r-q\right)}{\left(2-\beta\right)\delta^2\left[e^{(2-\beta)(r-q)(T-t)}-1\right]},\tag{12}$$

e

$$x:=\kappa S_t^{2-\beta}e^{(2-\beta)(r-q)(T-t)}.\tag{13}$$

Combinando as equações (10) e (11),

$$RA_t\left(S,X,T\right)=MS_te^{-q(T-t)}\left[F_{\chi^2\left(2+\frac{2}{2-\beta},2x\right)}\left(2\kappa X_b^{2-\beta}\right)-F_{\chi^2\left(2+\frac{2}{2-\beta},2x\right)}\left(2\kappa X_a^{2-\beta}\right)\right].\tag{14}$$

No caso em apreço

$$\delta=\frac{18\%}{10^{\frac{1.5-2}{2}}}\cong 0.32,$$

$$\kappa=\frac{2\left(3\%-0\%\right)}{\left(2-1.5\right)0.32^2\left[e^{(2-1.5)(3\%-0\%)\times 0.25}-1\right]}\cong 311.74,$$

e

$$x=311.74\times 10^{2-1.5}e^{(2-1.5)(3\%-0\%)\times 0.25}\cong 989.51.$$

Portanto,

$$\begin{aligned}RA_t&=100\times EUR10\\&\quad\times\left[F_{\chi^2(6,1979.02)}\left(2\times 311.74\times 12^{2-1.5}\right)-F_{\chi^2(6,1979.02)}\left(2\times 311.74\times 8^{0.5}\right)\right]\\&=EUR1,000\times\left[F_{\chi^2(6,1979.02)}\left(2159.79\right)-F_{\chi^2(6,1979.02)}\left(1763.46\right)\right].\end{aligned}$$

Utilizando a tabela do enunciado,

$$\begin{aligned}RA_t&=EUR1,000\times\left(0.9733455-0.005379885\right)\\&\cong EUR967.97.\end{aligned}$$

(a) O valor actual do IRS corresponde à diferença entre o valor actual do ramo variável (100%) menos o valor actual do ramo fixo:

$$
\begin{aligned}
IRS\,(0) &= 100\% - \left[\frac{6\%}{2} \times P\,(0,0.5) + 3\% \times P\,(0,1) + 103\% \times P\,(0,1.5)\right] \\
&= 100\% - (3\% \times 0.9826 + 3\% \times 0.9706 + 103\% \times 0.9627) \\
&\cong -5.022\%.
\end{aligned}
$$

Para um montante igual a 1,000,000 EUR, o valor do IRS é igual a **50,220 EUR**.

(b) O valor actual do *floorlet* pode ser escrito como uma call Europeia sobre uma obrigação de cupão zero:

$$
Floorlet_0 = e1M \times (1 + 0.5 \times 3\%) \times c_0 \left[P\,(0,1.5)\,; \frac{1}{1+0.5\times 3\%}; 1\right].
$$

Utilizando a Proposição 59 dos apontamentos,

$$
\begin{aligned}
& c_0 \left[P\,(0,1.5)\,; \frac{1}{1+0.5\times 3\%}; 1\right] \\
&= P\,(0,1.5)\,\Phi\left(d_1^V\right) - \frac{P\,(0,1)}{1+0.5\times 3\%}\Phi\left(d_0^V\right) \\
&= 0.9627 \times \Phi\left(d_1^V\right) - 0.9706 \times 0.9852 \times \Phi\left(d_0^V\right),
\end{aligned}
$$

onde

$$
\begin{aligned}
v\,(0,1,1.5) &= \sqrt{\frac{0.06^2}{1^2}\left[1 - e^{-1(1.5-1)}\right]^2 \frac{1-e^{-2\times 1\times 1}}{2\times 1}} \\
&\cong 1.552\%,
\end{aligned}
$$

$$
\begin{aligned}
d_1^V &= \frac{\ln\left(\frac{0.9627}{0.9706\times 0.9852}\right) + \frac{(1.552\%)^2}{2}}{1.552\%} \\
&\cong 0.443750107,
\end{aligned}
$$

e

$$
\begin{aligned}
d_0^V &= 0.443750107 - 1.552\% \\
&\cong 0.428227273.
\end{aligned}
$$

Portanto,

$$
\begin{aligned}
& c_0 \left[P\,(0,1.5)\,; \frac{1}{1+0.5\times 3\%}; 1\right] \\
&= 0.9627 \times \Phi\,(0.443750107) - 0.9706 \times 0.9852 \times \Phi\,(0.428227273) \\
&= 0.9627 \times 0.671388344 - 0.9706 \times 0.9852 \times 0.665757137 \\
&\cong 0.974\%, \qquad (15)
\end{aligned}
$$

e

$$
\begin{aligned}
Floorlet_0 &= e1M \times (1 + 0.5 \times 3\%) \times 0.974\% \\
&\cong e9,890.72.
\end{aligned}
$$

(c) De acordo com a Proposição 61 dos apontamentos, o valor actual da *call* sobre a CBB pode ser decomposto numa carteira de 2 *calls* Europeias sobre PBD:

$$\begin{aligned} & c_0 \left(B_t; X = 102.37\%; T = 1\right) \\ = \; & 2.5\% \times c_0 \left[P\left(0, 1.5\right); X_1; T = 1\right] + 102.5\% \times c_0 \left[P\left(0, 2\right); X_2; T = 1\right]. \end{aligned}$$

Os *strikes* podem ser obtidos via equação (327) dos apontamentos:

$$\begin{aligned} X_1 &= \exp\left[A\left(1.5 - 1\right) - B\left(1.5 - 1\right) \times 3.334\%\right] \\ &= \exp\left(-0.0018 - 0.3935 \times 3.334\%\right) \\ &\cong 98.52\%, \end{aligned}$$

e

$$\begin{aligned} X_2 &= \exp\left[A\left(2 - 1\right) - B\left(2 - 1\right) \times 3.334\%\right] \\ &= \exp\left(-0.0046 - 0.6321 \times 3.334\%\right) \\ &\cong 97.47\%. \end{aligned}$$

Portanto,

$$\begin{aligned} & c_0 \left(B_t; X = 102.37\%; T = 1\right) \\ = \; & 2.5\% \times c_0 \left[P\left(0, 1.5\right); 98.52\%; T = 1\right] + 102.5\% \times c_0 \left[P\left(0, 2\right); 97.47\%; T = 1\right]. \end{aligned} \tag{16}$$

A segunda *call* já foi avaliada na alínea anterior –vide equação (15):

$$c_0 \left[P\left(0, 1.5\right); \frac{1}{1 + 0.5 \times 3\%} = 0.9852; 1\right] \cong 0.974\%. \tag{17}$$

Relativamente à primeira *call*, o enunciado fornece o seguinte valor actual:

$$c_0 \left[P\left(0, 2\right); 97.47\%; T = 1\right] = 1.649\%. \tag{18}$$

Combinando as equações (16), (17) e (18),

$$\begin{aligned} & c_0 \left(B_t; X = 102.37\%; T = 1\right) \\ = \; & 2.5\% \times 0.974\% + 102.5\% \times 1.649\% \\ \cong \; & 1.715\%. \end{aligned}$$

(a) Visto que um caplet também pode ser avaliado como sendo uma put Europeia sobre uma obrigação de cupão zero, então

$$c_0 \left[E\left(0, 1, 1.5\right); 5\%; 1.5\right] = \frac{\left(1 + 5\% \times 0.5\right)}{0.5} \times p_0 \left[P\left(0, 1.5\right); \frac{1}{1 + 5\% \times 0.5}; 1\right]. \tag{19}$$

Via Proposição 68

$$p_0\left[P\left(0,1.5\right);0.9756;1\right] = -P\left(0,1.5\right)Q_{\chi^2_{\left(\frac{4\times2\times3.5\%}{0.1^2},\zeta_2\right)}}\left(\frac{r^*}{L_2}\right) +97.56\%\times P\left(0,1\right)Q_{\chi^2_{\left(28,\zeta_1\right)}}\left(\frac{r^*}{L_1}\right), \tag{20}$$

sendo

$$\begin{aligned}\gamma &= \sqrt{2^2+2\times\left(10\%\right)^2}\\ &\cong 2.004993766,\end{aligned}$$

$$\begin{aligned}\zeta_2 &= \frac{8r_t\gamma^2e^{\gamma(T_1-t)}}{\sigma^2\left[e^{\gamma(T_1-t)}-1\right]\left\{\gamma\left[e^{\gamma(T_1-t)}+1\right]+\left[k-\sigma^2B\left(T_2-T_1\right)\right]\left[e^{\gamma(T_1-t)}-1\right]\right\}}\\ &= \left[8\times4\%\times\left(2.004993766\right)^2\times e^{2.004993766\times1}\right]\Big\{0.1^2\times\left(e^{2.004993766\times1}-1\right)\\ &\quad 2.004993766\times\left(e^{2.004993766\times1}+1\right)\\ &\quad +\left(2-0.1^2\times\left(-0.3160\right)\right)\left(e^{2.004993766\times1}-1\right)\Big]\Big\}^{-1}\\ &\cong 4.994143104,\end{aligned}$$

$$\begin{aligned}L_2 &= \frac{\sigma^2}{2}\frac{e^{\gamma(T_1-t)}-1}{\gamma\left[e^{\gamma(T_1-t)}+1\right]+\left[k-\sigma^2B\left(T_2-T_1\right)\right]\left[e^{\gamma(T_1-t)}-1\right]}\\ &= \frac{\frac{0.1^2}{2}\times\left(e^{2.004993766}-1\right)}{2.004993766\times\left[e^{2.004993766}+1\right]+\left[2-0.1^2\times\left(-0.3160\right)\right]\left(e^{2.004993766\times1}-1\right)}\\ &\cong 0.001079407\end{aligned}$$

e

$$\begin{aligned}r^* &= \frac{\ln\left(K\right)-A\left(T_2-T_1\right)}{B\left(T_2-T_1\right)}\\ &= \frac{\ln\left(97.56\%\right)-A\left(1.5-1\right)}{B\left(0.5\right)}\\ &= \frac{\ln\left(97.56\%\right)-\left(-0.0064\right)}{-0.3160}\\ &\cong 5.777\%,\end{aligned}$$

Portanto,

$$p_0\left[P\left(0,1.5\right);0.9756;1\right] = -0.9466\times Q_{\chi^2_{(28,4.994143104)}}\left(\frac{5.777\%}{0.001079407}\right) +97.56\%\times0.9635\times Q_{\chi^2_{(28,4.997552141)}}\left(\frac{5.777\%}{0.001080143}\right). \tag{21}$$

As probabilidades contidas na equação anterior podem ser calculada via aproximação de Sankaran, i.e.

$$\begin{aligned} Q_{\chi^2(a,b)}(z) &= \mathbb{P}\left(\chi^2(a,b) \geq z\right) \\ &= \mathbb{P}\left\{\left[\frac{\chi^2(a,b)}{a+b}\right]^h \geq \left(\frac{z}{a+b}\right)^h\right\} \\ &\approx \Phi\left[-\frac{\left(\frac{z}{a+b}\right)^h - \mu_h}{\sigma_h}\right], \end{aligned} \tag{22}$$

onde

$$\mu_h := 1 + h(h-1)\frac{a+2b}{(a+b)^2} - h(h-1)(2-h)(1-3h)\frac{(a+2b)^2}{2(a+b)^4}, \tag{23}$$

$$\sigma_h^2 := h^2\frac{2(a+2b)}{(a+b)^2}\left[1-(1-h)(1-3h)\frac{a+2b}{(a+b)^2}\right], \tag{24}$$

e

$$h := 1 - \frac{2}{3}(a+b)(a+3b)(a+2b)^{-2}. \tag{25}$$

Começando por $Q_{\chi^2_{(28,4.994143104)}}(53.52433488)$, $a = 28$, $b = 4.997552141$,

$$\begin{aligned} h &= 1 - \frac{2}{3}(28 + 4.997552141)(28 + 3 \times 4.997552141) \\ &\quad (28 + 2 \times 4.997552141)^{-2} \\ &\cong 0.344855424, \end{aligned}$$

$$\begin{aligned} \mu_h &= 1 + 0.344855424 \times (0.344855424 - 1)\frac{28 + 2 \times 4.997552141}{(28 + 4.997552141)^2} \\ &\quad -0.344855424 \times (0.344855424 - 1)(2 - 0.344855424) \\ &\quad (1 - 3 \times 0.344855424)\frac{(28 + 2 \times 4.997552141)^2}{2(28 + 4.997552141)^4} \\ &\cong 0.992108065, \end{aligned}$$

e

$$\begin{aligned} \sigma_h^2 &: = 0.344855424^2 \times \frac{2(28 + 2 \times 4.997552141)}{(28 + 4.997552141)^2} \\ &\quad \left[1 - (1 - 0.344855424)(1 - 3 \times 0.344855424)\frac{28 + 2 \times 4.997552141}{(28 + 4.997552141)^2}\right] \\ &\cong 0.008306598. \end{aligned}$$

Utilizando a equação (22),

$$\begin{aligned} Q_{\chi^2_{(28,4.994143104)}}(53.52433488) &= \Phi\left[-\frac{\left(\frac{53.52433488}{28+4.997552141}\right)^{0.344855424} - 0.992108065}{\sqrt{0.008306598}}\right] \\ &\cong 0.01881887. \end{aligned} \tag{26}$$

Finalmente, combinando as equações (21) e (26),

$$\begin{aligned} p_0\left[P\left(0,1.5\right);0.9756;1\right] &= -0.9466\times 0.018979585 \\ &\quad +97.56\%\times 0.9635\times 0.01881887 \\ &\cong 0.00035213. \end{aligned} \tag{27}$$

Em suma, combinando as equações (19) e (27),

$$\begin{aligned} c_0\left[E\left(0,1,1.5\right);5\%;1.5\right] &= \frac{(1+5\%\times 0.5)}{0.5}\times 0.00035213 \\ &\cong 0.000721864. \end{aligned}$$

(b) Como o futuro é um $\mathbb{Q}$-martingale, então

$$F_0 = EUR50,000\times \mathbb{E}_{\mathbb{Q}}\left[100\% - E\left(T,T+0.5\right)\middle|\,\mathcal{F}_0\right].$$

Por outro lado, como

$$P\left(T,T+0.5\right) = \frac{1}{1+0.5\times E\left(T,T+0.5\right)},$$

então

$$\begin{aligned} F_0 &= EUR50,000\times \mathbb{E}_{\mathbb{Q}}\left\{1-\frac{1}{0.5}\left[\frac{1}{P\left(T,T+0.5\right)}-1\right]\middle|\,\mathcal{F}_0\right\} \\ &= EUR50,000\times \left\{1+2-\frac{1}{0.5}\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{P\left(T,T+0.5\right)}\middle|\,\mathcal{F}_0\right]\right\} \\ &= EUR50,000\times \left\{3-2\times \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-A\left(0.5\right)-B\left(0.5\right)r_T\right)\middle|\,\mathcal{F}_0\right]\right\} \\ &= EUR50,000\times \left\{3-2\times \exp\left(-A\left(0.5\right)\right)\mathbb{E}_{\mathbb{Q}}\left[\exp\left(-B\left(0.5\right)r_T\right)\middle|\,\mathcal{F}_0\right]\right\}. \end{aligned} \tag{28}$$

O valor esperado $\mathbb{E}_{\mathbb{Q}}\left[\exp\left(-B\left(0.5\right)r_T\right)\middle|\,\mathcal{F}_0\right]$ pode ser obtido via Proposição 66 com $\lambda = B\left(0.5\right)$, $\mu = 0$, $t = T$ e $t_0 = 0$:

$$\mathbb{E}_{\mathbb{Q}}\left[\exp\left(-B\left(0.5\right)r_T\right)\middle|\,\mathcal{F}_0\right] = \exp\left[\phi_{B(0.5),0}\left(T\right)-4\%\times\psi_{B(0.5),0}\left(T\right)\right]. \tag{29}$$

Para o contrato em apreço, $T=1$ e, portanto,

$$h = k = 2,$$

$$\begin{aligned} \phi_{B(0.5),0}\left(1\right) &= \frac{2\times 2\times 3.5\%}{0.1^2}\ln\left[\frac{2\times 2\times e^{\frac{4\times 1}{2}}}{0.1^2\times\left(-0.316\right)\left(e^2-1\right)+2-2+\left(2+2\right)e^2}\right] \\ &\cong 0.00956646, \end{aligned}$$

$$\begin{aligned} \psi_{B(0.5),0}\left(1\right) &= \frac{-0.316\times\left[2+2+\left(2-2\right)e^2\right]+2\times 0\times\left[e^2-1\right]}{0.1^2\times\left(-0.316\right)\left(e^2-1\right)+2-2+\left(2+2\right)e^2} \\ &\cong -0.04279518. \end{aligned}$$

Recuperando a equação (29),

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left[\exp\left(-B\left(0.5\right)r_T\right)\middle|\,\mathcal{F}_0\right] &= \exp\left[0.00956646 - 4\% \times \left(-0.04279518\right)\right] \\ &\cong 1.0113421,\end{aligned}$$

e, portanto, a equação (28) implica que

$$\begin{aligned}F_0 &= EUR50,000 \times \left[3 - 2 \times \exp\left(0.0064\right) \times 1.0113421\right] \\ &\cong EUR48,216.45.\end{aligned}$$

Referências