Modelos de Estrutura Temporal de Taxas de Juro
Mestrado em Matemática Financeira 11/12
IBS e FCUL
Exame 1ª Época - Resolução

18/Dez/12 Duração: 3h

1. (a) Based on Proposition 66 of the handouts, we know that

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-\lambda r_T}\exp\left(-\mu\int_t^T r_s ds\right)|\mathcal{F}_t\right] = \exp\left[\phi_{\lambda,\mu}(T-t) - r_t\psi_{\lambda,\mu}(T-t)\right], \quad (1)$$

where functions $\phi_{\lambda,\mu}(T-t)$ and $\psi_{\lambda,\mu}(T-t)$ are given by equations (363) and (364) of the handouts.

Hence, the moment generation function of r_T under measure $\mathbb{Q}$ is given by:

$$\begin{aligned} M(t,T;r,\theta) &= \mathbb{E}_{\mathbb{Q}}\left(e^{\theta r_T}\middle|\mathcal{F}_t\right) \\ &= \exp\left[\phi_{-\theta,0}(T-t) - r_t\psi_{-\theta,0}(T-t)\right], \quad (2) \end{aligned}$$

for any $\theta \in \mathbb{R}$. Consequently, the first moment of the random variable r_T can be found through the first derivative of its moment generation function, i.e.:

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}\left(r_T|\mathcal{F}_t\right) &= \left.\frac{\partial}{\partial\theta}M(t,T;r,\theta)\right|_{\theta=0} \\ &= \left\{\left[\frac{\partial}{\partial\theta}\phi_{-\theta,0}(T-t) - r_t\frac{\partial}{\partial\theta}\psi_{-\theta,0}(T-t)\right]M(t,T;r,\theta)\right\}_{\theta=0} \\ &= \left.\frac{\partial}{\partial\theta}\phi_{-\theta,0}(T-t)\right|_{\theta=0} - r_t\left.\frac{\partial}{\partial\theta}\psi_{-\theta,0}(T-t)\right|_{\theta=0}. \end{aligned}$$

(b) Using equation (152) of the handouts, the European-style call price is

$$c_t(S_t,X,T) = S_t e^{-q(T-t)}P_1(S_t,v_t;T,X) - e^{-r(T-t)}XP_2(S_t,v_t;T,X), \quad (3)$$

with

$$P_j(S_t,v_t;T,X) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{\exp(-i\phi\ln X)\varphi_j(S_t,v_t;T,\phi)}{i\phi}\right]d\phi, \quad (4)$$

for $j = 1, 2$, and where

$$\varphi_j(S_t,v_t;T,\phi) = \exp\left[C_j(\phi;T-t) + D_j(\phi;T-t)v_t + i\phi\ln S_t\right]. \quad (5)$$

Functions $C_j(\phi;T-t)$ and $D_j(\phi;T-t)$ are given by equations (135) and (136) of the handouts.

We want to show that

$$c_t\left(\lambda S_t, \lambda X, T\right) = \lambda c_t\left(S_t, X, T\right), \tag{6}$$

for $\lambda \in \mathbb{R}$. The key step is to show that the exercise probabilities $P_j\left(S_t, v_t; T, X\right)$ are homogeneous of degree zero in the spot and the strike. For this purpose, and combining equations (4) and (5), then

$$\begin{aligned}
& P_j\left(\lambda S_t, v_t; T, \lambda X\right) \\
= & \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}\left[\frac{\exp\left(-i\phi \ln\left(\lambda X\right)\right)\varphi_j\left(\lambda S_t, v_t; T, \phi\right)}{i\phi}\right] d\phi \\
= & \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}\left\{\frac{\exp\left(-i\phi \ln\left(\lambda X\right)\right)\exp\left[C_j\left(\phi; T-t\right) + D_j\left(\phi; T-t\right)v_t + i\phi\ln\left(\lambda S_t\right)\right]}{i\phi}\right\} d\phi \\
= & \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}\left[\frac{\exp\left(-i\phi \ln\left(X\right)\right)\varphi_j\left(S_t, v_t; T, \phi\right)}{i\phi}\right] d\phi \\
= & P_j\left(S_t, v_t; T, X\right).
\end{aligned} \tag{7}$$

Combining equations (3) and (7), then

$$\begin{aligned}
c_t\left(\lambda S_t, \lambda X, T\right) &= \lambda S_t e^{-q(T-t)} P_1\left(\lambda S_t, v_t; T, \lambda X\right) - e^{-r(T-t)}\lambda X P_2\left(\lambda S_t, v_t; T, \lambda X\right) \\
&= \lambda S_t e^{-q(T-t)} P_1\left(S_t, v_t; T, X\right) - e^{-r(T-t)}\lambda X P_2\left(S_t, v_t; T, X\right) \\
&= \lambda\left[S_t e^{-q(T-t)} P_1\left(S_t, v_t; T, X\right) - e^{-r(T-t)} X P_2\left(S_t, v_t; T, X\right)\right] \\
&= \lambda c_t\left(S_t, X, T\right).
\end{aligned}$$

2. Via equação (64) dos handouts, o valor da call Europeia (com $\beta > 2$) é dado por:

$$c_t\left(S, X, T\right) = S_t e^{-q(T-t)} Q_{\chi^2\left(\frac{2}{\beta-2}, 2\kappa X^{2-\beta}\right)}\left(2x\right) - X e^{-r(T-t)}\left[1 - Q_{\chi^2\left(2+\frac{2}{\beta-2}, 2x\right)}\left(2\kappa X^{2-\beta}\right)\right], \tag{8}$$

$$\kappa := \frac{2\left(r-q\right)}{\left(2-\beta\right)\delta^2\left[e^{(2-\beta)(r-q)(T-t)} - 1\right]}, \tag{9}$$

e

$$x := \kappa S_t^{2-\beta} e^{(2-\beta)(r-q)(T-t)}. \tag{10}$$

Retomando as equações (9) e (10),

$$\kappa = \frac{2\left(1\% - 3\%\right)}{\left(2-4\right)\left(0.04\right)^2\left[e^{(2-4)(1\%-3\%)\times 2} - 1\right]} \cong 150.0833244,$$

e

$$x = 150.0833244 \times \left(5\right)^{2-4} e^{(2-4)(1\%-3\%)\times 2} \cong 6.503332978.$$

(a) Substituindo na equação (8), então

$$\begin{aligned}
c_t &= 5 \times e^{-3\%\times 2} \times Q_{\chi^2\left(\frac{2}{4-2}, 2\times 150.0833244\times 5^{2-4}\right)}\left(2 \times 6.503332978\right) \\
&\quad -5 \times e^{-1\%\times 2} \times \left[1 - Q_{\chi^2\left(2+\frac{2}{4-2}, 2\times 6.503332978\right)}\left(2 \times 150.0833244 \times 5^{2-4}\right)\right] \\
&= 5 \times e^{-3\%\times 2} \times Q_{\chi^2(1, 12.00666596)}\left(13.00666596\right) \\
&\quad -5 \times e^{-1\%\times 2} \times F_{\chi^2(3, 13.00666596)}\left(12.00666596\right).
\end{aligned} \tag{11}$$

Via tabela do enunciado, sabemos que

$$F_{\chi^2(3,13.00666596)}(12.00666596) = 0.333949115. \tag{12}$$

A probabilidade $Q_{\chi^2(1,12.00666596)}(13.00666596)$ pode ser calculada via aproximação de Sankaran, i.e.

$$\begin{aligned} Q_{\chi^2(a,b)}(z) &= 1 - \mathbb{Q}\left(\chi^2(a,b) < z\right) \\ &= 1 - \mathbb{Q}\left\{\left[\frac{\chi^2(a,b)}{a+b}\right]^h < \left(\frac{z}{a+b}\right)^h\right\} \\ &\approx \Phi\left[-\frac{\left(\frac{z}{a+b}\right)^h - \mu_h}{\sigma_h}\right], \end{aligned} \tag{13}$$

onde

$$\mu_h := 1 + h(h-1)\frac{a+2b}{(a+b)^2} - h(h-1)(2-h)(1-3h)\frac{(a+2b)^2}{2(a+b)^4}, \tag{14}$$

$$\sigma_h^2 := h^2\frac{2(a+2b)}{(a+b)^2}\left[1-(1-h)(1-3h)\frac{a+2b}{(a+b)^2}\right], \tag{15}$$

e

$$h := 1 - \frac{2}{3}(a+b)(a+3b)(a+2b)^{-2}. \tag{16}$$

Visto que $a = 1$ e $b = 12.00666596$, então

$$\begin{aligned} h &= 1 - \frac{2}{3}(1+12.00666596)(1+3\times 12.00666596) \\ &\quad (1+2\times 12.00666596)^{-2} \\ &\cong 0.486940156, \end{aligned}$$

$$\begin{aligned} \mu_h &= 1 + 0.486940156 \times (0.4869401568 - 1)\frac{1+2\times 12.00666596}{(1+12.00666596)^2} \\ &\quad -0.486940156 \times (0.486940156 - 1)(2 - 0.486940156) \\ &\quad (1 - 3\times 0.486940156)\frac{(1+2\times 12.00666596)^2}{2(1+12.00666596)^4} \\ &\cong 0.961157106, \end{aligned}$$

e

$$\begin{aligned} \sigma_h^2 &= 0.486940156^2 \times \frac{2(1+2\times 12.00666596)}{(1+12.00666596)^2} \\ &\quad \left[1-(1-0.486940156)(1-3\times 0.486940156)\frac{1+2\times 12.00666596}{(1+12.00666596)^2}\right] \\ &\cong 0.072567679. \end{aligned}$$

Utilizando a equação (13),

$$\Phi\left[-\frac{\left(\frac{z}{a+b}\right)^h-\mu_h}{\sigma_h}\right]$$

$$\begin{aligned}
Q_{\chi^2(1,12.00666596)}(13.00666596) &= \Phi\left[-\frac{\left(\frac{13.00666596}{1+12.00666596}\right)^{0.486940156}-0.961157106}{\sqrt{0.072567679}}\right] \\
&\cong 0.442674586. \qquad (17)
\end{aligned}$$

Finalmente, combinando as equações (11), (12) e (17),

$$\begin{aligned}
c_t &= 5\times e^{-3\%\times 2}\times 0.442674586 - 5\times e^{-1\%\times 2}\times 0.333949115 \\
&\cong EUR0.44779.
\end{aligned}$$

3. (a) Utilizando a Proposição 22 dos apontamentos,

$$\begin{aligned}
c_0 &= 5\times e^{-3\%\times 0.25}\times P_1\left(S_t=5, v_t=0.04; T=0.25, X=5\right) \qquad (18) \\
&\quad -e^{-1\%\times 0.25}\times 5\times P_2\left(S_t=5, v_t=0.04; T=0.25, X=5\right).
\end{aligned}$$

Com base nas equações (173) e (174) dos apontamentos:

$$\begin{aligned}
P_1\left(S_t=5, v_t=0.04; T=0.25, X=5\right) &\approx \frac{1}{2}+\frac{0.04412373}{\pi} \\
&\cong 0.5140450, \qquad (19)
\end{aligned}$$

e

$$\begin{aligned}
P_2\left(S_t=5, v_t=0.04; T=0.25, X=5\right) &\approx \frac{1}{2}+\frac{-0.08197835}{\pi} \\
&\cong 0.4739055. \qquad (20)
\end{aligned}$$

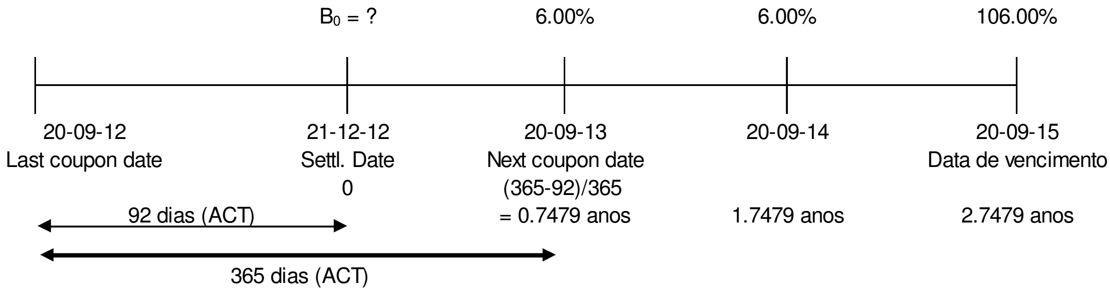
Combinando as equações (18), (19) e (20),

$$\begin{aligned}
c_0 &= 5\times e^{-3\%\times 0.25}\times 0.5140450 - e^{-1\%\times 0.25}\times 5\times 0.4739055 \\
&\cong EUR0.18741.
\end{aligned}$$

(b) Com base na alínea b) do Caso 1, sabemos que no modelo de Heston (1993) uma *call* Europeia é uma função homógenea de grau 1 no *spot* e no *strike*. Consequentemente,

$$\begin{aligned}
c_0\left(S_0=10; X=10; T=0.25\right) &= 2\times c_0\left(S_0=5; X=5; T=0.25\right) \\
&= 2\times 0.18741 \\
&= EUR0.37482.
\end{aligned}$$

4. (a) Settlement date = 18/12/12 + 3 dias de calendário = 21/12/12.
Pretende-se avaliar uma obrigação com os seguintes cash flows vincendos:



Portanto,

$$\begin{aligned} B_0 &= 6\% \times P(0, 0.7479) + 6\% \times P(0, 1.7479) + 106\% \times P(0, 2.7479) \\ &= 6\% \times 0.9939 + 6\% \times 0.9843 + 106\% \times P(0, 2.7479). \end{aligned} \tag{21}$$

Relativamente ao factor de desconto a 2.7479 anos, via equações (268), (269) e (270) dos apontamentos,

$$\begin{aligned} B(2.7479) &= \frac{1 - e^{-4\times 2.7479}}{4} \\ &\cong 0.25, \end{aligned}$$

$$\begin{aligned} A(2.7479) &= (0.25 - 2.7479)\left(0.01 - \frac{0.1^2}{2 \times 4^2}\right) - \frac{0.1^2}{4 \times 4} \times (0.25)^2 \\ &\cong -0.0242, \end{aligned}$$

e

$$\begin{aligned} P(0, 2.7479) &= \exp(-0.0242 - 0.25 \times 0.5\%) \\ &\cong 0.9748. \end{aligned}$$

Retomando a equação (21), então

$$\begin{aligned} B_0 &= 6\% \times 0.9939 + 6\% \times 0.9843 + 106\% \times 0.9748 \\ &\cong 115.201\%. \end{aligned}$$

(b) Montante de juros vencidos:

$$\begin{aligned} AI &= 6\% \times \frac{92}{365} \\ &\cong 1.512\%. \end{aligned}$$

Portanto,

$$VT^{bid} = 113.50\% + 1.512\% = 115.012\%.$$

Visto que $VT^{bid} < B_0$, então decide-se não vender. Por outro lado,

$$VT^{ask} = 113.60\% + 1.512\% = 115.112\%.$$

Visto que $VT^{ask} < B_0$, então decide-se comprar.

(c) Sendo a opção ATM-forward, então o *strike* é dado por:

$$X = \frac{P(0,2.5)}{P(0,1)} = \frac{0.9772}{0.9915} \cong 0.9856.$$

Utilizando a Proposição 59 dos apontamentos,

$$\begin{aligned} & p_0\left[P(0,2.5);98.56\%;1\right] \\ = & -P(0,2.5)\,\Phi\left(-d_1^V\right) + 98.56\% \times P(0,1)\,\Phi\left(-d_0^V\right) \\ = & -0.9772 \times \Phi\left(-d_1^V\right) + 98.56\% \times 0.9915 \times \Phi\left(-d_0^V\right), \end{aligned}$$

onde

$$\begin{aligned} v(0,1,2.5) &= \sqrt{\frac{0.1^2}{4^2}\left[1-e^{-4\times(2.5-1)}\right]^2 \frac{1-e^{-2\times 4\times 1}}{2\times 4}} \\ &\cong 0.882\%, \end{aligned}$$

$$\begin{aligned} d_1^V &= \frac{\ln\left(\frac{0.9772}{0.9856\times 0.9915}\right) + \frac{(0.882\%)^2}{2}}{0.882\%} \\ &\cong 0.004407723, \end{aligned}$$

e

$$\begin{aligned} d_0^V &= 0.004407723 - 0.882\% \\ &\cong -0.004407723. \end{aligned}$$

Portanto,

$$\begin{aligned} & p_0\left[P(0,2.5);98.56\%;1\right] \\ = & -0.9772 \times \Phi(-0.004407723) + 98.56\% \times 0.9915 \times \Phi(-0.004407723) \\ = & -0.9772 \times 0.498241579 + 98.56\% \times 0.9915 \times 0.501758421 \\ \cong & \ 0.344\%. \end{aligned} \tag{22}$$

5. Pretende-se avaliar a seguinte opção:

$$p_0\left[P(0,3);K=97.182\%;1\right].$$

Via Proposição 68,

$$\begin{aligned} p_0\left[P(0,3);K=97.182\%;1\right] &= -P(0,3) \times Q_{\chi^2_{\left(\frac{4\times 2\times 1\%}{0.1^2},\zeta_2\right)}}\left(\frac{r^*}{L_2}\right) \\ &\quad +0.97182 \times P(0,1) \times Q_{\chi^2_{(8,\zeta_1)}}\left(\frac{r^*}{L_1}\right), \end{aligned} \tag{23}$$

sendo

$$\begin{aligned} \gamma &= \sqrt{2^2 + 2\times(10\%)^2} \\ &\cong 2.004993766, \end{aligned}$$

$$
\begin{aligned}
\zeta_2 &= \frac{8r_t\gamma^2 e^{\gamma(T_1-t)}}{\sigma^2\left[e^{\gamma(T_1-t)}-1\right]\left\{\gamma\left[e^{\gamma(T_1-t)}+1\right]+\left[k-\sigma^2 B\left(T_2-T_1\right)\right]\left[e^{\gamma(T_1-t)}-1\right]\right\}} \\
&= \left[8\times 1\%\times(2.004993766)^2\times e^{2.004993766\times 1}\right]\Big\{0.1^2\times\left(e^{2.004993766\times 1}-1\right) \\
&\quad \Big[2.004993766\times\left(e^{2.004993766\times 1}+1\right) \\
&\quad +\left(2-0.1^2\times(-0.4903)\right)\left(e^{2.004993766\times 1}-1\right)\Big]\Big\}^{-1} \\
&\cong 1.248066071,
\end{aligned}
$$

$$
\begin{aligned}
L_2 &= \frac{\sigma^2}{2}\frac{e^{\gamma(T_1-t)}-1}{\gamma\left[e^{\gamma(T_1-t)}+1\right]+\left[k-\sigma^2 B\left(T_2-T_1\right)\right]\left[e^{\gamma(T_1-t)}-1\right]} \\
&= \frac{\frac{0.1^2}{2}\times\left(e^{2.004993766\times 1}-1\right)}{2.004993766\left(e^{2.004993766\times 1}+1\right)+\left(2-0.1^2\left(-0.4903\right)\right)\left(e^{2.004993766\times 1}-1\right)} \\
&\cong 0.001079001,
\end{aligned}
$$

e

$$
\begin{aligned}
r^* &= \frac{\ln(K)-A\left(T_2-T_1\right)}{B\left(T_2-T_1\right)} \\
&= \frac{\ln(0.97182)-A(3-1)}{B(2)} \\
&= \frac{\ln(0.97182)-(-0.0151)}{-0.4903} \\
&\cong 2.753\%.
\end{aligned}
$$

Portanto,

$$
\begin{aligned}
& p_0\left[P(0,3); K=97.182\%; 1\right] \\
=\; & -0.9705\times Q_{\chi^2_{(8,1.248066071)}}\left(\frac{2.753\%}{0.001079001}\cong 25.51724263\right) \\
& +0.97182\times 0.9901\times Q_{\chi^2_{(8,1.2493880)}}\left(\frac{2.753\%}{0.0010801}\cong 25.49024311\right).
\end{aligned}
\tag{24}
$$

(a) Com base no enunciado, podemos calcular as duas probabilidades contidas na equação anterior:

$$
Q_{\chi^2_{(8,1.248066071)}}(25.51724263) = 1-0.995478682 = 0.004521318, \tag{25}
$$

e

$$
Q_{\chi^2_{(8,1.2493880)}}(25.49024311) = 1-0.995433822 = 0.004566178. \tag{26}
$$

Finalmente, combinando as equações (24), (25) e (26),

$$
\begin{aligned}
p_0\left[P(0,3); K=97.182\%; 1\right] &= -0.9705\times 0.004521318 \\
&\quad +0.97182\times 0.9901\times 0.004566178 \\
&\cong 0.00000557.
\end{aligned}
$$

Referências

Heston, S., 1993, A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, *Review of Financial Studies* 6, 327–343.