Modelos de Estrutura Temporal de Taxas de Juro
Mestrado em Matemática Financeira 14/15
IBS e FCUL
Exame 1ª Época

18/Dez/15 Duração: 3h

Case 1 Please answer only two of the following questions: (2x3V)

a) Under the Cox, Ingersoll and Ross (1985) model, compute the (time t) *fair price* of a futures contract with expiry date at time T, with a *contract size* equal to M, and on the Euribor $E(T, T+\delta)$ to prevail between times T $(\geq t)$ and $T+\delta$ (with $\delta > 0$). For this purpose, please remember that a futures contract is a $\mathbb{Q}$-martingale, that $P(T, T+\delta) = [1 + \delta \times E(T, T+\delta)]^{-1}$ and that the terminal *payoff* of the future contract under analysis is

$$F_T = M \times \delta \times [100\% - E(T, T+\delta)].$$

b) Proposition 61 of the handouts shows that, under the Vasiček (1977) model, a European-style option on a coupon-bearing bond can be decomposed into a portfolio of options on zero-coupon bonds, with the same expiry date and with adjusted strikes. Please adjust Proposition 61 to the context of the Cox et al. (1985) model, and show that the main result still holds.

c) Using the Vasiček (1977) model and under the *risk-neutral probability measure* $\mathbb{Q}$, please compute the variance of the random variable $\int_{t_0}^{t} r_s ds$, for $t_0 \leq t$ and conditional to $\mathcal{F}_{t_0}$.

Case 2 Consider a CEV process given by the following SDE:

$$dS_t = (r - q) S_t dt + \delta S_t^{\frac{\beta}{2}} dW_t^{\mathbb{Q}}.$$

Assume that $S = \$10$, $\beta = 1$, $r = 1\%$, $q = 2\%$ and that the (annualized) standard deviation of stock returns is equal to 20%. Please consider also the following table containing cumulative probabilities associated to a noncentral chi-square random variable with 2 degrees of freedom and a noncentrality parameter equal to 400.50021:

x	399.50021	400.50021	499.50020
F(x)	0.48006	0.49004	0.98965

Please answer the following questions:

a) Price an ATM call on the asset S and with a time-to-maturity of 0.25 years. (2V)

b) Compute the integral $\int_{400.50021}^{\infty} f_{\chi^2(4,b)}(399.50021)\, db$, where $f_{\chi^2(a,b)}(x)$ is the density function of a noncentral chi-square random variable with a degrees of freedom and a noncentrality parameter equal to b. (1V)

Case 3 Consider the following parameters for the Heston (1993) model:

- Spot price of the GN stock = EUR15;
- *Dividend yield* for the GN stock (continuous compounding) = 2% (30/360);
- Risk-free interest rate (continuous compounding) = 1% (30/360);
- Instantaneous variance of the stock returns (v) = 0.09;
- Speed of mean reversion of the volatility (k) = 2;
- Long-term level of the instantaneous variance (θ) = 0.06;
- Volatility of the instantaneous variance (σ) = 10%; and
- Correlation coefficient between the stock price and the instantaneous variance (ρ) = -0.4.

Next table summarizes the implementation of equations (173) and (174) of the handouts for the strikes EUR10 and EUR20, for a maturity of 6 months, and through a Gauss-Laguerre quadrature with 15 nodes:

w_i	ϕ_i	$X = 10$		$X = 20$	
		$f_1(\phi_i)$	$f_2(\phi_i)$	$f_1(\phi_i)$	$f_2(\phi_i)$
2.1823E-01	9.3308E-02	4.6094E-01	4.1780E-01	-2.9971E-01	-3.4287E-01
3.4221E-01	4.9269E-01	6.7946E-01	6.1657E-01	-4.4347E-01	-5.0682E-01
2.6303E-01	1.2156E+00	1.3180E+00	1.2034E+00	-8.7786E-01	-9.9796E-01
1.2643E-01	2.2699E+00	3.1495E+00	2.9319E+00	-2.2359E+00	-2.5011E+00
4.0207E-02	3.6676E+00	8.2216E+00	8.0613E+00	-6.8958E+00	-7.4218E+00
8.5639E-03	5.4253E+00	1.7628E+01	2.0287E+01	-2.3527E+01	-2.3253E+01
1.2124E-03	7.5659E+00	-8.2472E+00	1.5990E+01	-7.6794E+01	-6.1704E+01
1.1167E-04	1.0120E+01	-3.2795E+02	-2.5206E+02	-1.7216E+02	-4.4867E+01
6.4599E-06	1.3130E+01	-6.4585E+02	-1.1049E+03	1.9603E+02	7.9582E+02
2.2263E-07	1.6654E+01	5.5908E+03	3.4150E+03	4.1002E+03	5.0839E+03
4.2274E-09	2.0776E+01	-8.1925E+03	4.3241E+03	1.8491E+04	1.3776E+04
3.9219E-11	2.5624E+01	1.0472E+04	-2.0307E+04	4.3195E+04	1.6521E+04
1.4565E-13	3.1408E+01	-5.4169E+04	-6.2229E+02	6.0945E+04	1.1297E+04
1.4830E-16	3.8531E+01	-1.1290E+02	2.9653E+04	4.9435E+04	1.6313E+04
1.6006E-20	4.8026E+01	-1.3399E+04	-3.3849E+03	3.9365E+03	8.3940E+03
$\sum_{i=1}^{15} w_i f_j(\phi_i) =$		1.5099E+00	1.47210171	-1.31955534	-1.39799306

Please answer the following questions:

a) Price a European-style put on the GN stock, with a strike price equal to EUR10, and with a time-to-maturity of 0.5 years. (1V)

b) Price a *range cash-or-nothing option* on the GN stock, with a time-to-maturity of 0.5 years, a contract size equal to EUR100, and *strikes* equal to EUR10 and EUR20. (1V)

Case 4 Consider the following parameters, estimated under measure $\mathbb{Q}$, for the Vasiček (1977) model, using German treasury bonds for the settlement date of 18/12/15:

alpha	3
gamma	1%
rho	10%
r(t)	1.0%

Next table shows discount factors (for different maturities) based on the previous model' parameters:

T-t	B(t,T)	A(t,T)	P(t,T)
0.5	0.2590	-0.0023	0.9951
1	0.3167	-0.0065	0.9903
1.4044	0.3284	-0.0103	0.9866
2	0.3325	-0.0158	0.9810
2.4044	0.3331	-0.0197	0.9773
3	0.3333	-0.0253	0.9718

Please answer the following questions:

a) Find the fair value of a treasury bond with maturity at 14/05/2018, with *bullet* redemption, and with a coupon rate of 2% (annual coupon under the daycount convention ACT/ACT). The first (long) coupon will be paid on 14/05/2016 but the first coupon period started on 19/09/2014. Consider that the number of calendar days between 19/09/2014 and 14/05/2015 is equal to 237 days, and the number of calendar days between 14/05/2015 and 18/12/2015 is equal to 218 days. (2V)

b) Price a ATM European-style call, with maturity at 14/05/2016 and on a Treasury Bill with maturity at 14/05/2017. (2V)

Case 5 Consider the following parameters, estimated under measure $\mathbb{Q}$ and using IRS quotes, for the Cox et al. (1985):

k	4.0
theta	3.0%
sigma	5.0%
r	1.0%

Next table contains discount factors (for different maturities) based on the previous model' parameters:

T-t	B(T-t)	A(T-t)	P(t,T)
0.5	-0.21615758	-0.00851483	0.98938039
1	-0.24540443	-0.02263638	0.97522172
1.5	-0.24936136	-0.03751652	0.96077970
2	-0.24989671	-0.05249929	0.94648680
2.5	-0.24996914	-0.06749595	0.93239789
3	-0.24997894	-0.08249448	0.91851755

Consider also the following table containing cumulative probabilities associated to a noncentral chi-square random variable with 192 degrees of freedom and a noncentrality parameter equal to b:

	F(x)		
x	97.5729719	97.5729719	97.5804516
b			
1.19349762	6.92383E-12	1.32720E-09	1.33206E-09
1.19358911	6.92366E-12	1.32717E-09	1.33203E-09

The market also trades European-style options with a time-to-maturity of one year, and on zero-coupon bonds (with money-market credit risk and) with a time-to-maturity of two years:

	strike	94.531%	97.403%
Call		2.460%	0.000%
Put		0.000%	0.341%

Please answer the following questions:

a) Price a European-style put with a time-to-maturity of one year, with a strike equal to 94.531%, and on a zero-coupon bond with a time-to-maturity of three years, knowing that $L_1 = 0.000153378$ and $\zeta_1 = 1.193589112$. (2V)

b) Price a European-style put with a time-to-maturity of one year, with a strike equal to 98.37%, and on a coupon-bearing bond with a time-to-maturity of three years, an annual coupon rate of 2% and bullet redemption. For this purpose, consider that an instantaneous interest rate of 1.497% yields, after one year, a fair value of 98.37% for the underlying coupon-bearing bond. (3V)

Referências

Cox, J., J. Ingersoll, and S. Ross, 1985, A Theory of the Term Structure of Interest Rates, *Econometrica* 53, 385–407.

Heston, S., 1993, A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, *Review of Financial Studies* 6, 327–343.

Vasiček, O., 1977, An Equilibrium Characterization of the Term Structure, *Journal of Financial Economics* 5, 177–188.