Modelos de Estrutura Temporal de Taxas de Juro
Mestrado em Matemática Financeira 09/10
IBS e FCUL
Exame 1ª Época - Resolução

15/Dez/09 Duração: 3h

1. (a) Como é hábito,

$$\begin{aligned}
P(t,T) &= \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s ds\right)|\mathcal{F}_t\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left\{\exp\left[-\int_t^T\left(\sum_{j=1}^n Y_j(s)\right)ds\right]|\mathcal{F}_t\right\} \\
&= \mathbb{E}_{\mathbb{Q}}\left\{\exp\left[\sum_{j=1}^n\left(-\int_t^T Y_j(s)\,ds\right)\right]|\mathcal{F}_t\right\} \\
&= \mathbb{E}_{\mathbb{Q}}\left\{\prod_{j=1}^n \exp\left[-\int_t^T Y_j(s)\,ds\right]|\mathcal{F}_t\right\}.
\end{aligned}$$

Visto que as variáveis de estado Y_j são independentes entre si, então

$$P(t,T) = \prod_{j=1}^n \mathbb{E}_{\mathbb{Q}}\left\{\exp\left[-\int_t^T Y_j(s)\,ds\right]|\mathcal{F}_t\right\}.$$

Por outro lado, cada variavel de estado Y_j segue um square-root process similar ao adoptado pelo modelo CIR, a saber:

$$dr_t = k(\theta - r_t)\,dt + \sigma\sqrt{r_t}dW_t^{\mathbb{Q}}.$$

Consequentemente,

$$\exp\left[-\int_t^T Y_j(s)\,ds\right] = \exp\left(-\int_t^T r_s ds\right),$$

com $k = k_j$, $\theta = \theta_j$, $\sigma = \sigma_j$ e $r_s = Y_j(s)$. Então, utilizando a Proposição 64 dos apontamentos,

$$P(t,T) = \prod_{j=1}^n \exp\left[A_j(T-t) + B_j(T-t)\,Y_j(t)\right],$$

sendo

$$B_j(\tau) = \frac{-2\left(e^{\gamma_j\tau}-1\right)}{2\gamma_j + \left(k_j+\gamma_j\right)\left(e^{\gamma_j\tau}-1\right)},$$

e

$$A_j(\tau) = \frac{2k_j\theta_j}{\sigma_j^2} \ln \left[\frac{2\gamma_j e^{\left(k_j+\gamma_j\right)\frac{\tau}{2}}}{2\gamma_j + \left(k_j+\gamma_j\right)\left(e^{\gamma_j \tau}-1\right)} \right],$$

com

$$\tau := T - t,$$

e

$$\gamma_j := \sqrt{k_j^2 + 2\sigma_j^2}.$$

(b) Começando pelo payoff terminal,

$$\begin{aligned} V_T &= (S_T - X_1)\, \mathbb{1}_{\{S_T > X_2\}} \\ &= \left[(S_T - X_2) + (X_2 - X_1)\right] \mathbb{1}_{\{S_T > X_2\}} \\ &= (S_T - X_2)\, \mathbb{1}_{\{S_T > X_2\}} + (X_2 - X_1)\, \mathbb{1}_{\{S_T > X_2\}} \\ &= c_T(S, X_2, T) + (X_2 - X_1)\, \mathbb{1}_{\{S_T > X_2\}}. \end{aligned}$$

Portanto,

$$V_t = c_t(S, X_2, T) + \mathbb{E}_{\mathbb{Q}}\left[(X_2 - X_1)\, \mathbb{1}_{\{S_T > X_2\}} | \mathcal{F}_t\right]. \tag{1}$$

A call standard $c_t(S, X_2, T)$ pode ser avaliada via equação (58) dos apontamentos, i.e.

$$\begin{aligned} & c_t(S_t, X, T) \\ = \; & S_t e^{-q(T-t)} Q_{\chi^2\left(2+\frac{2}{2-\beta}, 2x\right)}\left(2\kappa X^{2-\beta}\right) - X e^{-r(T-t)}\left[1 - Q_{\chi^2\left(\frac{2}{2-\beta}, 2\kappa X^{2-\beta}\right)}(2x)\right], \end{aligned} \tag{2}$$

sendo

$$\kappa := \frac{2(r-q)}{(2-\beta)\,\delta^2\left[e^{(2-\beta)(r-q)(T-t)} - 1\right]}, \tag{3}$$

e

$$x := \kappa S_t^{2-\beta} e^{(2-\beta)(r-q)(T-t)}. \tag{4}$$

Relativamente ao segundo termo no lado direito da equação (1), o mesmo corresponde ao valor de uma *cash-or-nothing* call com *contract size* igual a $(X_2 - X_1)$ e strike igual a X_2:

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}}\left[(X_2 - X_1)\, \mathbb{1}_{\{S_T > X_2\}} | \mathcal{F}_t\right] \\ = \; & (X_2 - X_1)\, \mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{S_T > X_2\}} | \mathcal{F}_t\right] \\ = \; & (X_2 - X_1)\, \mathbb{Q}\left(S_T > X_2 | \mathcal{F}_t\right). \end{aligned}$$

Utilizando a Proposição 13 dos apontamentos,

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}}\left[(X_2 - X_1)\, \mathbb{1}_{\{S_T > X_2\}} | \mathcal{F}_t\right] \\ = \; & (X_2 - X_1)\left[1 - Q_{\chi^2\left(\frac{2}{2-\beta}, 2\kappa S_T^{2-\beta}\right)}(2x)\right]. \end{aligned}$$

(c) No modelo de Vasiček (1977),

$$dr_t = \alpha \left(\gamma - r_t\right) dt + \rho dW_t^{\mathbb{Q}}. \tag{5}$$

De forma similar ao descrito na demonstração da Proposição 66, é possível mostrar que

$$F\left(t - t_0, r_{t_0}\right) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\lambda r_t} \exp\left(-\mu \int_{t_0}^{t} r_s ds\right) |\mathcal{F}_{t_0}\right] \tag{6}$$

é a solução da seguinte EDP

$$\frac{\partial F}{\partial t_0} - \alpha\left(\gamma - r_{t_0}\right)\frac{\partial F}{\partial r_{t_0}} - \frac{1}{2}\rho^2 \frac{\partial^2 F}{\partial r_{t_0}^2} + \mu r_{t_0} F = 0, \tag{7}$$

sujeita à condição terminal

$$F\left(0, r_{t_0}\right) = e^{-\lambda r_{t_0}}. \tag{8}$$

Substituindo a *trial solution*

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-\lambda r_t} \exp\left(-\mu \int_{t_0}^{t} r_s ds\right) |\mathcal{F}_{t_0}\right] = \exp\left[\phi_{\lambda,\mu}\left(t - t_0\right) - r_{t_0}\psi_{\lambda,\mu}\left(t - t_0\right)\right] \tag{9}$$

na EDP (7), obtém-se

$$\begin{aligned} 0 &= \left[\frac{\partial \phi\left(t - t_0\right)}{\partial t_0} - r_{t_0}\frac{\partial \psi\left(t - t_0\right)}{\partial t_0}\right] F + \alpha\left(\gamma - r_{t_0}\right)\psi\left(t - t_0\right) F \\ &\quad -\frac{1}{2}\rho^2\psi^2\left(t - t_0\right) F + \mu r_{t_0} F \\ &= \left[\frac{\partial \phi\left(t - t_0\right)}{\partial t_0} + \alpha\gamma\psi\left(t - t_0\right) - \frac{1}{2}\rho^2\psi^2\left(t - t_0\right)\right] \\ &\quad + \left[-\frac{\partial \psi\left(t - t_0\right)}{\partial t_0} - \alpha\psi\left(t - t_0\right) + \mu\right] r_{t_0}. \end{aligned}$$

Consequentemente, $\psi\left(t - t_0\right)$ e $\phi\left(t - t_0\right)$ são obtidas via solução das duas seguintes ODEs

$$\frac{\partial \psi\left(t - t_0\right)}{\partial t_0} = -\alpha\psi\left(t - t_0\right) + \mu, \tag{10}$$

e

$$\frac{\partial \phi\left(t - t_0\right)}{\partial t_0} = -\alpha\gamma\psi\left(t - t_0\right) + \frac{1}{2}\rho^2\psi^2\left(t - t_0\right), \tag{11}$$

sujeitas às condições terminais

$$\psi\left(0\right) = \lambda, \tag{12}$$

e

$$\phi\left(0\right) = 0. \tag{13}$$

Relativamente à ODE (10), e visto que $\frac{\partial\psi(t-t_0)}{\partial t_0} = -\frac{\partial\psi(t-t_0)}{\partial t}$,

$$\frac{\partial\psi}{\alpha\psi-\mu} = \partial t,$$

i.e.

$$\frac{1}{\alpha}\ln\left(\alpha\psi-\mu\right) = t + C, \tag{14}$$

sendo C a constante de integração. Utilizando a condição terminal (12),

$$\frac{1}{\alpha}\ln\left(\alpha\lambda-\mu\right) = t_0 + C,$$

i.e.

$$C = \frac{1}{\alpha}\ln\left(\alpha\lambda-\mu\right) - t_0. \tag{15}$$

Combinando as equações (14) e (15),

$$\begin{aligned}
\frac{1}{\alpha}\ln\left(\alpha\psi-\mu\right) &= t + \frac{1}{\alpha}\ln\left(\alpha\lambda-\mu\right) - t_0 \\
\ln\left(\frac{\alpha\psi-\mu}{\alpha\lambda-\mu}\right) &= \alpha\left(t-t_0\right) \\
\psi_{\lambda,\mu}\left(t-t_0\right) &= \frac{\left(\alpha\lambda-\mu\right)e^{\alpha(t-t_0)}+\mu}{\alpha}.
\end{aligned} \tag{16}$$

Relativamente à função $\phi_{\lambda,\mu}\left(t-t_0\right)$, combinando as equações (11) e (13), e visto que $\frac{\partial\phi_{\lambda,\mu}(t-t_0)}{\partial t_0} = -\frac{\partial\phi_{\lambda,\mu}(t-t_0)}{\partial t}$, então

$$\begin{aligned}
&\phi_{\lambda,\mu}\left(t-t_0\right) \\
&= \alpha\gamma\int_{t_0}^{t}\psi\left(s-t_0\right)ds - \frac{1}{2}\rho^2\int_{t_0}^{t}\psi^2\left(s-t_0\right)ds \\
&= \alpha\gamma\int_{t_0}^{t}\frac{\left(\alpha\lambda-\mu\right)e^{\alpha(s-t_0)}+\mu}{\alpha}ds - \frac{1}{2}\rho^2\int_{t_0}^{t}\left[\frac{\left(\alpha\lambda-\mu\right)e^{\alpha(s-t_0)}+\mu}{\alpha}\right]^2 ds.
\end{aligned} \tag{17}$$

Em sintese, a solução final é dada pelas equações (9), (16) e (17).

(a) Utilizando a Proposição 22 dos apontamentos,

$$\begin{aligned}
c_0 &= 10\times e^{-2\%\times 1}\times P_1\left(S_t=10, v_t=0.03; T=1, X=10\right) \\
&\quad -e^{-1\%\times 1}\times 10\times P_2\left(S_t=10, v_t=0.03; T=1, X=10\right).
\end{aligned} \tag{18}$$

Com base nas equações (173) e (174) dos apontamentos:

$$\begin{aligned}
P_1\left(S_t=10, v_t=0.03; T=1, X=10\right) &\approx \frac{1}{2}+\frac{0.11358604}{\pi} \\
&\cong 5.3616E-01,
\end{aligned} \tag{19}$$

e

$$
\begin{aligned}
P_2\left(S_t=10, v_t=0.03; T=1, X=10\right) &\approx \frac{1}{2}+\frac{-0.08043763}{\pi} \\
&\cong 4.7440E-01. \qquad (20)
\end{aligned}
$$

Combinando as equações (18), (19) e (20),

$$
\begin{aligned}
c_0 &= 10 \times e^{-2\% \times 1} \times (5.3616E-01) - e^{-1\% \times 1} \times 10 \times (4.7440E-01) \\
&\cong EUR0.55863.
\end{aligned}
$$

(b) O payoff terminal de uma opção range asset-or-nothing é dado por

$$
RA_T\left(S, X, T\right) = M S_T \mathbb{1}_{\{X_a < S_T < X_b\}}.
$$

Consequentemente,

$$
\begin{aligned}
RA_t\left(S, X_a, X_b, T\right) &= e^{-r(T-t)} M \mathbb{E}_{\mathbb{Q}}\left(S_T \mathbb{1}_{\{X_a<S_T<X_b\}} \middle| \mathcal{F}_t\right) \\
&= S_t e^{qt} M \mathbb{E}_{\mathbb{Q}_S}\left(\frac{S_T \mathbb{1}_{\{X_a<S_T<X_b\}}}{S_T e^{qT}} \middle| \mathcal{F}_t\right) \\
&= M S_t e^{-q(T-t)} \mathbb{Q}_S\left(X_a < S_T < X_b \middle| \mathcal{F}_t\right) \\
&= M S_t e^{-q(T-t)} \left[\mathbb{Q}_S\left(S_T < X_b \middle| \mathcal{F}_t\right) - \mathbb{Q}_S\left(S_T < X_a \middle| \mathcal{F}_t\right)\right] \qquad (21)
\end{aligned}
$$

Visto que

$$
\mathbb{Q}_S\left(S_T < X \middle| \mathcal{F}_t\right) = 1 - P_1\left(S_t, v_t; T, X\right), \qquad (22)
$$

então combinando as equações (21) e (22),

$$
RA_t\left(S, X_a, X_b, T\right) = M S_t e^{-q(T-t)} \left[P_1\left(S_t, v_t; T, X_a\right) - P_1\left(S_t, v_t; T, X_b\right)\right]. \qquad (23)
$$

No caso em apreço

$$
\begin{aligned}
RA_0 &= 1 \times 10 \times e^{-2\% \times 1} \times \left[P_1\left(S_t=10, v_t=0.03; T=1, X=8\right)\right. \\
&\quad \left. -P_1\left(S_t=10, v_t=0.03; T=1, X=10\right)\right] \\
&= 1 \times 10 \times e^{-2\% \times 1} \times \left[\frac{1}{2}+\frac{1.32071319}{\pi} - (5.3616E-01)\right] \\
&= 1 \times 10 \times e^{-2\% \times 1} \times \left[(9.2040E-01) - (5.3616E-01)\right] \\
&\cong EUR3.766.
\end{aligned}
$$

(a) Pretende-se calcular

$$
P\left(3,5\right) = A\left(3,5\right) \exp\left[-B\left(3,5\right) \times 3\%\right]. \qquad (24)
$$

A função $B(3,5)$ é calculada tal como no modelo de Vasiček (1977):

$$
\begin{aligned}
B\left(3,5\right) &= \frac{1-e^{-0.2\times(5-3)}}{0.2} \\
&\cong 1.64839977. \qquad (25)
\end{aligned}
$$

A função $A(3,5)$ é dada por

$$\begin{aligned}
& \ln A(3,5) \\
= \; & \ln\left[\frac{P(0,5)}{P(0,3)}\right] - B(3,5)\left.\frac{\partial \ln P(0,t)}{\partial t}\right|_{t=3} \\
& + \frac{0.05^2}{4\times 0.2^3}\left(e^{-0.2\times 5} - e^{-0.2\times 3}\right)^2\left(1 - e^{2\times 0.2\times 3}\right) \\
\approx \; & \ln\left[\frac{0.853824048}{0.934561116}\right] - 1.64839977 \times \frac{\ln P(0,3.01) - \ln P(0,2.99)}{2\times 0.01} \\
& + \frac{0.05^2}{4\times 0.2^3}\left(e^{-0.2\times 5} - e^{-0.2\times 3}\right)^2\left(1 - e^{2\times 0.2\times 3}\right) \\
= \; & \ln\left[\frac{0.853824048}{0.934561116}\right] - 1.64839977 \times \frac{\ln(0.934196128) - \ln(0.934925529)}{2\times 0.01} \\
& + \frac{0.05^2}{4\times 0.2^3}\left(e^{-0.2\times 5} - e^{-0.2\times 3}\right)^2\left(1 - e^{2\times 0.2\times 3}\right) \\
\cong \; & -0.031958971. \qquad (26)
\end{aligned}$$

Combinando as equações (24), (25) e (26), então

$$\begin{aligned}
P(3,5) &= \exp\left[-0.031958971 - 1.64839977 \times 3\%\right] \\
&\cong 0.921814781.
\end{aligned}$$

(b) O valor actual da call Europeia sobre a obrigação de cupão zero é dado por

$$c_0\left[P(0,5); 92.334\%; 3\right] = P(0,5)\times\Phi\left(d_1^{HW}\right) - 92.334\% \times P(0,3)\times\Phi\left(d_0^{HW}\right).$$

Utilizando a Proposição 59 dos apontamentos, mas usando os factores de desconto em vigor no mercado, então

$$\begin{aligned}
v(0,3,5) &= \sqrt{\frac{0.05^2}{0.2^2}\left[1 - e^{-0.2\times(5-3)}\right]^2 \frac{1 - e^{-2\times 0.2\times 3}}{2\times 0.2}} \\
&\cong 10.894\%,
\end{aligned}$$

$$\begin{aligned}
d_1^{HW} &= \frac{\ln\left[\frac{P(0,5)}{92.334\%\times P(0,3)}\right] + \frac{(10.894\%)^2}{2}}{10.894\%} \\
&= \frac{\ln\left(\frac{0.853824048}{92.334\%\times 0.934561116}\right) + \frac{(10.894\%)^2}{2}}{10.894\%} \\
&\cong -0.042744958,
\end{aligned}$$

e

$$\begin{aligned}
d_0^{HW} &= -0.042744958 - 10.894\% \\
&\cong -0.15168331.
\end{aligned}$$

Portanto,

$$\begin{aligned}
& c_0\left[P\left(0,5\right);92.334\%;3\right] \\
= \;& 0.853824048 \times \Phi\left(-0.042744958\right) \\
& -92.334\% \times 0.934561116 \times \Phi\left(-0.15168331\right) \\
= \;& 0.853824048 \times 0.482952356 - 92.334\% \times 0.934561116 \times 0.439718343 \\
\cong \;& 3.292\%. \qquad (27)
\end{aligned}$$

(c) De acordo com a Proposição 61 dos apontamentos, mas usando os factores de desconto em vigor no mercado, o valor actual da *call* sobre a CBB pode ser decomposto numa carteira de 2 *calls* Europeias sobre PBD:

$$\begin{aligned}
& c_0\left(B_t; X = 98\%; T = 3\right) \\
= \;& 3\% \times c_0\left[P\left(0,4\right); X_1; T = 3\right] + 103\% \times c_0\left[P\left(0,5\right); X_2; T = 3\right].
\end{aligned}$$

Os *strikes* podem ser obtidos via equação (327) dos apontamentos:

$$\begin{aligned}
X_1 &= \exp\left[\ln A\left(3,4\right) - B\left(3,4\right) \times 2.9\%\right] \\
&= \exp\left(-0.008880834 - 0.906346235 \times 2.9\%\right) \\
&\cong 96.545\%,
\end{aligned}$$

e

$$\begin{aligned}
X_2 &= \exp\left[\ln A\left(3,5\right) - B\left(3,5\right) \times 2.9\%\right] \\
&= \exp\left(-0.031958971 - 1.64839977 \times 2.9\%\right) \\
&\cong 92.334\%.
\end{aligned}$$

Portanto,

$$\begin{aligned}
& c_0\left(B_t; X = 98\%; T = 3\right) \qquad (28) \\
= \;& 3\% \times c_0\left[P\left(0,4\right); 96.545\%; T = 3\right] + 103\% \times c_0\left[P\left(0,5\right); 92.334\%; T = 3\right].
\end{aligned}$$

A segunda *call* já foi avaliada na alínea anterior –vide equação (27):

$$c_0\left[P\left(0,5\right); 92.334\%; T = 3\right] \cong 3.292\%. \qquad (29)$$

Relativamente à primeira *call*, o enunciado fornece o seguinte valor actual:

$$c_0\left[P\left(0,4\right); 96.545\%; T = 3\right] = 1.836\%. \qquad (30)$$

Combinando as equações (28), (29) e (30),

$$\begin{aligned}
& c_0\left(B_t; X = 98\%; T = 3\right) \\
= \;& 3\% \times 1.836\% + 103\% \times 3.292\% \\
\cong \;& 3.4455\%.
\end{aligned}$$

(a) Pretende-se avaliar a seguinte opção:

$$p_0\left[P(0,5);\frac{P(0,5)}{P(0,2)};2\right]=p_0\left[P(0,5)=0.8723;\frac{0.8723}{0.9538}\cong 0.9146;2\right]. \tag{31}$$

Via Proposição 68

$$\begin{aligned} p_0\left[P(0,5);\frac{P(0,5)}{P(0,2)};2\right] &= -P(0,5)\,Q_{\chi^2_{\left(\frac{4\times 1.5\times 3\%}{0.05^2},\zeta_2\right)}}\left(\frac{r^*}{L_2}\right) \\ &\quad +0.9146\times P(0,2)\,Q_{\chi^2_{(72,\zeta_1)}}\left(\frac{r^*}{L_1}\right), \end{aligned} \tag{32}$$

sendo

$$\begin{aligned} \gamma &= \sqrt{1.5^2+2\times(5\%)^2} \\ &\cong 1.501665742, \end{aligned}$$

$$\begin{aligned} \zeta_2 &= \frac{8r_t\gamma^2e^{\gamma(T_1-t)}}{\sigma^2\left[e^{\gamma(T_1-t)}-1\right]\left\{\gamma\left[e^{\gamma(T_1-t)}+1\right]+\left[k-\sigma^2B(T_2-T_1)\right]\left[e^{\gamma(T_1-t)}-1\right]\right\}} \\ &= \left[8\times 1\%\times(1.501665742)^2\times e^{1.501665742\times 2}\right]\left\{0.05^2\times\left(e^{1.501665742\times 2}-1\right)\right. \\ &\quad 1.501665742\times\left(e^{1.501665742\times 2}+1\right) \\ &\quad \left.+\left(1.5-0.05^2\times(-0.6589)\right)\left(e^{1.501665742\times 2}-1\right)\right]\Big\}^{-1} \\ &\cong 1.254494856, \end{aligned}$$

$$\begin{aligned} L_2 &= \frac{\sigma^2}{2}\frac{e^{\gamma(T_1-t)}-1}{\gamma\left[e^{\gamma(T_1-t)}+1\right]+\left[k-\sigma^2B(T_2-T_1)\right]\left[e^{\gamma(T_1-t)}-1\right]} \\ &= \frac{\frac{0.05^2}{2}\times\left(e^{1.501665742\times 2}-1\right)}{1.501665742\left(e^{1.501665742\times 2}+1\right)+\left(1.5-0.05^2(-0.6589)\right)\left(e^{1.501665742\times 2}-1\right)} \\ &\cong 0.000395554 \end{aligned}$$

e

$$\begin{aligned} r^* &= \frac{\ln(K)-A(T_2-T_1)}{B(T_2-T_1)} \\ &= \frac{\ln(0.9146)-A(5-2)}{B(3)} \\ &= \frac{\ln(0.9146)-(-0.0702)}{-0.6589} \\ &\cong 2.898\%, \end{aligned}$$

Portanto,

$$\begin{aligned} & p_0\left[P(0,5);\frac{P(0,5)}{P(0,2)};2\right] \\ &= -0.8723\times Q_{\chi^2_{(72,1.254494856)}}\left(\frac{2.898\%}{0.000395554}\right) \\ &\quad +0.9146\times 0.9538\times Q_{\chi^2_{(72,1.255149144)}}\left(\frac{2.898\%}{0.00039576}\cong 73.27392186\right). \end{aligned} \tag{33}$$

A segunda probabilidade contidas na equação anterior é dada já no enunciado, i.e.

$$Q_{\chi^2_{(72,1.255149144)}}(73.27392186) = 0.477223473. \tag{34}$$

A primeira probabilidade pode ser calculada via aproximação de Sankaran, i.e.

$$\begin{aligned} Q_{\chi^2(a,b)}(z) &= \mathbb{P}\left(\chi^2(a,b) \geq z\right) \\ &= \mathbb{P}\left\{\left[\frac{\chi^2(a,b)}{a+b}\right]^h \geq \left(\frac{z}{a+b}\right)^h\right\} \\ &\approx \Phi\left[-\frac{\left(\frac{z}{a+b}\right)^h - \mu_h}{\sigma_h}\right], \end{aligned} \tag{35}$$

onde

$$\mu_h := 1 + h(h-1)\frac{a+2b}{(a+b)^2} - h(h-1)(2-h)(1-3h)\frac{(a+2b)^2}{2(a+b)^4}, \tag{36}$$

$$\sigma_h^2 := h^2\frac{2(a+2b)}{(a+b)^2}\left[1-(1-h)(1-3h)\frac{a+2b}{(a+b)^2}\right], \tag{37}$$

e

$$h := 1 - \frac{2}{3}(a+b)(a+3b)(a+2b)^{-2}. \tag{38}$$

Para calcular $Q_{\chi^2_{(72,1.254494856)}}(73.2357254)$, temos que $a = 72$, $b = 1.254494856$,

$$\begin{aligned} h &= 1 - \frac{2}{3}(72 + 1.254494856)(72 + 3 \times 1.254494856) \\ &\quad (72 + 2 \times 1.254494856)^{-2} \\ &\cong 0.33352251, \end{aligned}$$

$$\begin{aligned} \mu_h &= 1 + 0.33352251 \times (0.33352251 - 1)\frac{72 + 2 \times 1.254494856}{(72 + 1.254494856)^2} \\ &\quad -0.33352251 \times (0.33352251 - 1)(2 - 0.33352251) \\ &\quad (1 - 3 \times 0.33352251)\frac{(72 + 2 \times 1.254494856)^2}{2(72 + 1.254494856)^4} \\ &\cong 0.996913591, \end{aligned}$$

e

$$\begin{aligned} \sigma_h^2 &: = 0.33352251^2 \times \frac{2(72 + 2 \times 1.254494856)}{(72 + 1.254494856)^2} \\ &\quad \left[1 - (1 - 0.33352251)(1 - 3 \times 0.33352251)\frac{72 + 2 \times 1.254494856}{(72 + 1.254494856)^2}\right] \\ &\cong 0.003089033. \end{aligned}$$

Utilizando a equação (35),

$$
\begin{aligned}
Q_{\chi^2_{(72,1.254494856)}}(73.2357254) &= \Phi\left[-\frac{\left(\frac{73.2357254}{72+1.254494856}\right)^{0.33352251}-0.996913591}{\sqrt{0.003089033}}\right] \\
&\cong 0.478491186. \qquad (39)
\end{aligned}
$$

Finalmente, combinando as equações (33), (34) e (39),

$$
\begin{aligned}
p_0\left[P(0,5);\frac{P(0,5)}{P(0,2)};2\right] &= -0.8723\times 0.478491186 \\
&\quad +0.9146\times 0.9538\times 0.477223473 \\
&\cong 0.00110583.
\end{aligned}
$$

(b) O payoff terminal da opção, daqui a 2 anos (T), é dado por

$$
V_T = M\mathbb{1}_{\{E(T,T+1)>k\}},
$$

sendo M o contract size (EUR1,000,000) e k o strike. Por outro lado, como

$$
P(T,T+1) = \frac{1}{1+E(T,T+1)},
$$

então

$$
\begin{aligned}
V_T &= M\mathbb{1}_{\left\{P(T,T+1)<(1+k)^{-1}\right\}} \\
&= M\mathbb{1}_{\left\{\exp[A(1)+B(1)r_T]<(1+k)^{-1}\right\}} \\
&= M\mathbb{1}_{\left\{r_T>\frac{\ln(1+k)^{-1}-A(1)}{B(1)}\right\}}.
\end{aligned}
$$

Consequentemente, o valor actual da opção é dado por

$$
\begin{aligned}
V_t &= P(t,T)\,\mathbb{E}_{\mathbb{Q}_T}\left[M\mathbb{1}_{\left\{r_T>\frac{\ln(1+k)^{-1}-A(1)}{B(1)}\right\}}\middle|\mathcal{F}_t\right] \\
&= P(t,T)\,M\mathbb{Q}_T\left[r_T>\frac{\ln(1+k)^{-1}-A(1)}{B(1)}\middle|\mathcal{F}_t\right] \\
&= P(t,T)\,MQ_{\chi^2\left(\frac{4k\theta}{\sigma^2},\zeta\right)}\left[\frac{\ln(1+k)^{-1}-A(1)}{B(1)\,L}\right], \qquad (40)
\end{aligned}
$$

visto que

$$
\frac{r_T}{L}\overset{\mathbb{Q}_T}{\sim}\chi^2\left(\frac{4k\theta}{\sigma^2},\zeta\right),
$$

com

$$
\begin{aligned}
\zeta &= \frac{8r_t\gamma^2e^{\gamma(T-t)}}{\sigma^2\left[e^{\gamma(T-t)}-1\right]\left\{\gamma\left[e^{\gamma(T-t)}+1\right]+\left[k-\sigma^2\times 0\right]\left[e^{\gamma(T-t)}-1\right]\right\}} \\
&= \zeta_1 \\
&= 1.255149144,
\end{aligned}
$$

e

$$\begin{aligned}
L &= \frac{\sigma^2}{2}\frac{e^{\gamma(T-t)}-1}{\gamma\left[e^{\gamma(T-t)}+1\right]+\left[k-\sigma^2\times 0\right]\left[e^{\gamma(T-t)}-1\right]} \\
&= L_1 \\
&= 0.00039576.
\end{aligned}$$

Portanto,

$$\begin{aligned}
& Q_{\chi^2\left(\frac{4k\theta}{\sigma^2},\zeta\right)}\left[\frac{\ln\left(1+k\right)^{-1}-A\left(1\right)}{B\left(1\right)L}\right] \\
= & Q_{\chi^2\left(\frac{4\times 1.5\times 3\%}{0.05^2},1.255149144\right)}\left[\frac{\ln\left(1+2.992\%\right)^{-1}-\left(-0.0145\right)}{-0.5178\times 0.00039576}\right] \\
= & Q_{\chi^2(72,1.255149144)}\left[\frac{-\ln\left(1.02992\right)-\left(-0.0145\right)}{-0.5178\times 0.00039576}\right] \\
= & Q_{\chi^2(72,1.255149144)}\left(73.27392\right) \\
= & 0.477223473.
\end{aligned}$$

Finalmente, recuperando a equação (40), então

$$\begin{aligned}
V_t &= P\left(0,2\right)\times EUR1,000,000\times 0.477223473 \\
&= 0.9538\times EUR1,000,000\times 0.477223473 \\
&\cong EUR455,168.24.
\end{aligned}$$

Referências

Vasiček, O., 1977, An Equilibrium Characterization of the Term Structure, *Journal of Financial Economics* 5, 177–188.