Modelos de Estrutura Temporal de Taxas de Juro
Mestrado em Matemática Financeira 15/16
IBS e FCUL
Exame 1ª Época - Resolução

21/Dec/16 Duration: 3h

1. (a) The terminal payoff of a *cash-or-nothing put* with expiry date at time $T+\delta$, with a *contract size* equal to M, with a *strike* equal to k, and on the Euribor rate $E(T, T+\delta)$ to prevail between times T $(\geq t)$ and $T+\delta$ (with $\delta > 0$) is equal to:

$$V_{T+\delta} = M\mathbb{1}_{\{E(T,T+\delta)<k\}}. \tag{1}$$

Using the identity

$$P(T, T+\delta) = [1+\delta \times E(T, T+\delta)]^{-1},$$

then equation (1) can be rewritten as

$$\begin{aligned} V_{T+\delta} &= M\mathbb{1}_{\{P^{-1}(T,T+\delta)-1<k\delta\}} \\ &= M\mathbb{1}_{\{P^{-1}(T,T+\delta)<1+k\delta\}} \\ &= M\mathbb{1}_{\left\{P(T,T+\delta)>(1+k\delta)^{-1}\right\}}. \end{aligned}$$

Consequently, and using the *forward measure* $\mathbb{Q}_{T+\delta}$ associated to the numeraire $P(t, T+\delta)$, then

$$\begin{aligned} V_t &= P(t, T+\delta)\,\mathbb{E}_{\mathbb{Q}_{T+\delta}}\left(\left.\frac{M\mathbb{1}_{\left\{P(T,T+\delta)>(1+k\delta)^{-1}\right\}}}{P(T+\delta, T+\delta)}\right| \mathcal{F}_t\right) \\ &= MP(t, T+\delta)\,\mathbb{Q}_{T+\delta}\left[P(T, T+\delta) > (1+k\delta)^{-1}\middle|\, \mathcal{F}_t\right]. \end{aligned} \tag{2}$$

Using equation (342) of the handouts, then equation (2) yields

$$\begin{aligned} V_t &= MP(t, T+\delta)\,\mathbb{Q}_{T+\delta}\left[\exp(A(\delta) + B(\delta)\, r_T) > (1+k\delta)^{-1}\middle|\, \mathcal{F}_t\right] \\ &= MP(t, T+\delta)\,\mathbb{Q}_{T+\delta}\left[r_T < \left.\frac{\ln(1+k\delta)^{-1} - A(\delta)}{B(\delta)}\right| \mathcal{F}_t\right], \end{aligned} \tag{3}$$

since $B(\delta) < 0$.

Equation (378), (379) and (394) of the handouts show that

$$\mathbb{E}_{\mathbb{Q}_{T+\delta}}\left[\exp\left(-\lambda\frac{r_T}{L_{T+\delta}}\right)|\mathcal{F}_t\right] = \frac{\exp\left(-\frac{\lambda}{1+2\lambda}\zeta_{T+\delta}\right)}{(1+2\lambda)^{\frac{2k\theta}{\sigma^2}}}, \tag{4}$$

where

$$\zeta_{T+\delta} := \frac{8 r_t \gamma^2 e^{\gamma(T-t)}}{\sigma^2 \left[e^{\gamma(T-t)} - 1\right] \left\{\gamma \left[e^{\gamma(T-t)} + 1\right] + \left[k - \sigma^2 B(\delta)\right] \left[e^{\gamma(T-t)} - 1\right]\right\}}, \tag{5}$$

and

$$L_j := \frac{\sigma^2}{2} \frac{e^{\gamma(T-t)} - 1}{\gamma \left[e^{\gamma(T-t)} + 1\right] + \left[k - \sigma^2 B(\delta)\right] \left[e^{\gamma(T-t)} - 1\right]}. \tag{6}$$

Therefore, equation (3) becomes

$$\begin{aligned} V_t &= M P(t, T+\delta) \mathbb{Q}_{T+\delta} \left[\frac{r_T}{L_{T+\delta}} < \frac{\ln(1+k\delta)^{-1} - A(\delta)}{B(\delta) L_{T+\delta}} \middle| \mathcal{F}_t \right] \\ &= M P(t, T+\delta) F_{\chi^2_{\left(\frac{4k\theta}{\sigma^2} \zeta_{T+\delta}\right)}} \left(\frac{\ln(1+k\delta)^{-1} - A(\delta)}{B(\delta) L_{T+\delta}} \right). \end{aligned}$$

(b) The terminal payoff of an *asset-or-nothing call* with expiry date at time T, with a *contract size* equal to M, with a *strike* equal to K, and on a zero-coupon bond with expiry date at time $T+\delta$ (with $\delta > 0$) is equal to:

$$V_T = M P(T, T+\delta) \mathbb{1}_{\{P(T,T+\delta) > K\}}. \tag{7}$$

Using the *forward measure* $\mathbb{Q}_{T+\delta}$ associated to the numeraire $P(t, T+\delta)$, then

$$\begin{aligned} V_t &= P(t, T+\delta) \mathbb{E}_{\mathbb{Q}_{T+\delta}} \left(\frac{M P(T, T+\delta) \mathbb{1}_{\{P(T,T+\delta) > K\}}}{P(T, T+\delta)} \middle| \mathcal{F}_t \right) \\ &= M P(t, T+\delta) \mathbb{Q}_{T+\delta} \left(P(T, T+\delta) > K \middle| \mathcal{F}_t \right) \\ &= M P(t, T+\delta) \mathbb{Q}_{T+\delta} \left(P(T, T, T+\delta) > K \middle| \mathcal{F}_t \right). \end{aligned} \tag{8}$$

Based on equation (321) of the handouts, we know that

$$P(T, T, T+\delta) = P(t, T, T+\delta) \exp\left[\frac{1}{2} v^2(t, T, T+\delta) - Y^{\mathbb{Q}_{T+\delta}}\right], \tag{9}$$

where

$$Y^{\mathbb{Q}_{T+\delta}} \middle| \mathcal{F}_t \sim N^1\left(0, v^2(t, T, T+\delta)\right). \tag{10}$$

Combining equations (8), (9) and (10), then

$$\begin{aligned} V_t &= M P(t, T+\delta) \mathbb{Q}_{T+\delta} \left\{ P(t, T, T+\delta) \exp\left[\frac{1}{2} v^2(t, T, T+\delta) - Y^{\mathbb{Q}_{T+\delta}}\right] > K \middle| \mathcal{F}_t \right\} \\ &= M P(t, T+\delta) \mathbb{Q}_{T+\delta} \left\{ -Y^{\mathbb{Q}_{T+\delta}} > \ln\left[\frac{K}{P(t, T, T+\delta)}\right] - \frac{1}{2} v^2(t, T, T+\delta) \middle| \mathcal{F}_t \right\} \\ &= M P(t, T+\delta) \mathbb{Q}_{T+\delta} \left\{ Y^{\mathbb{Q}_{T+\delta}} < \ln\left[\frac{P(t, T, T+\delta)}{K}\right] + \frac{1}{2} v^2(t, T, T+\delta) \middle| \mathcal{F}_t \right\} \\ &= M P(t, T+\delta) \Phi \left\{ \frac{\ln\left[\frac{P(t,T,T+\delta)}{K}\right] + \frac{1}{2} v^2(t, T, T+\delta)}{v(t, T, T+\delta)} \right\}. \end{aligned}$$

(c) Starting with the terminal payoff,

$$V_T = MS_T \mathbb{1}_{\{S_T>K\}}.$$

Therefore,

$$V_t = Me^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left[S_T \mathbb{1}_{\{S_T>K\}}|\mathcal{F}_t\right]. \tag{11}$$

Changing to the EMM $\mathbb{Q}_S$ (associated to the numeraire $S_t e^{qt}$), then equation (11) becomes

$$\begin{aligned} V_t &= MS_t e^{qt}\mathbb{E}_{\mathbb{Q}_S}\left[\frac{S_T \mathbb{1}_{\{S_T>K\}}}{S_T e^{qT}}|\mathcal{F}_t\right] \\ &= MS_t e^{-q(T-t)}\mathbb{E}_{\mathbb{Q}_S}\left[\mathbb{1}_{\{S_T>K\}}|\mathcal{F}_t\right] \\ &= MS_t e^{-q(T-t)}\mathbb{Q}_S\left[S_T > K|\mathcal{F}_t\right]. \end{aligned} \tag{12}$$

Using, for instance, equations (64) and (106) of the handouts, equation (12) yields

$$V_t = MS_t e^{-q(T-t)} Q_{\chi^2\left(\frac{2}{\beta-2}, 2\kappa K^{2-\beta}\right)}(2x),$$

where

$$\kappa := \frac{2(r-q)}{(2-\beta)\,\delta^2\left[e^{(2-\beta)(r-q)(T-t)}-1\right]}, \tag{13}$$

and

$$x := \kappa S_t^{2-\beta} e^{(2-\beta)(r-q)(T-t)}. \tag{14}$$

(a) Using equation (59) of the handouts, the fair value of the European-style put (with $\beta < 2$) is given by:

$$p_t(S,X,T) = -S_t e^{-q(T-t)} F_{\chi^2\left(2+\frac{2}{2-\beta},2x\right)}\left(2\kappa X^{2-\beta}\right) + Xe^{-r(T-t)} Q_{\chi^2\left(\frac{2}{2-\beta},2\kappa X^{2-\beta}\right)}(2x), \tag{15}$$

where

$$\kappa := \frac{2(r-q)}{(2-\beta)\,\delta^2\left[e^{(2-\beta)(r-q)(T-t)}-1\right]}, \tag{16}$$

and

$$x := \kappa S_t^{2-\beta} e^{(2-\beta)(r-q)(T-t)}. \tag{17}$$

Since the (annualized) standard deviation of stock returns is equal to 30% per year, then, and using equation (2) of the handouts,

$$\delta = \frac{30\%}{(5)^{\frac{-3-2}{2}}} = 16.770510.$$

Using equations (16) and (17),

$$\kappa = \frac{2(1\%-3\%)}{(2+3)(16.77051)^2\left[e^{(2+3)(1\%-3\%)\times 0.5}-1\right]} \cong 0.00058323,$$

and

$$x = 0.00058323 \times (5)^{2+3} e^{(2+3)(1\%-3\%)\times 0.5} \cong 1.733703688.$$

Hence, equation (15) yields

$$\begin{aligned} p_t &= -5 \times e^{-3\%\times 0.5} \times F_{\chi^2\left(2+\frac{2}{2+3}, 2\times 1.733703688\right)} \left(2 \times 0.00058323 \times 5^{2+3}\right) \\ &\quad +5 \times e^{-1\%\times 0.5} \times Q_{\chi^2\left(\frac{2}{2+3}, 2\times 0.00058323\times 5^{2+3}\right)} \left(2 \times 1.733703688\right) \\ &= -5 \times e^{-3\%\times 0.5} \times F_{\chi^2(2.4,3.467407377)} \left(3.645185154\right) \\ &\quad +5 \times e^{-1\%\times 0.5} \times Q_{\chi^2(0.4,3.645185154)} \left(3.467407377\right). \end{aligned} \tag{18}$$

From the table provided in the exam, we know that

$$F_{\chi^2(2.4,3.467407377)} \left(3.645185154\right) = 0.361833. \tag{19}$$

The probability $Q_{\chi^2(0.4,3.645185154)} \left(3.467407377\right)$ can be computed using the Sankaran approximation, i.e.

$$\begin{aligned} Q_{\chi^2(a,b)} \left(z\right) &= 1 - \mathbb{Q}\left(\chi^2\left(a,b\right) < z\right) \\ &= 1 - \mathbb{Q}\left\{\left[\frac{\chi^2\left(a,b\right)}{a+b}\right]^h < \left(\frac{z}{a+b}\right)^h\right\} \\ &\approx \Phi\left[-\frac{\left(\frac{z}{a+b}\right)^h - \mu_h}{\sigma_h}\right], \end{aligned} \tag{20}$$

where

$$\mu_h := 1 + h\left(h-1\right)\frac{a+2b}{\left(a+b\right)^2} - h\left(h-1\right)\left(2-h\right)\left(1-3h\right)\frac{\left(a+2b\right)^2}{2\left(a+b\right)^4}, \tag{21}$$

$$\sigma_h^2 := h^2 \frac{2\left(a+2b\right)}{\left(a+b\right)^2}\left[1 - \left(1-h\right)\left(1-3h\right)\frac{a+2b}{\left(a+b\right)^2}\right], \tag{22}$$

and

$$h := 1 - \frac{2}{3}\left(a+b\right)\left(a+3b\right)\left(a+2b\right)^{-2}. \tag{23}$$

Since $a = 0.4$ e $b = 3.645185154$, then

$$\begin{aligned} h &= 1 - \frac{2}{3}\left(0.4 + 3.645185154\right)\left(0.4 + 3 \times 3.645185154\right) \\ &\quad \left(0.4 + 2 \times 3.6451851543\right)^{-2} \\ &\cong 0.483113194, \end{aligned}$$

$$\begin{aligned} \mu_h &= 1 + 0.483113194 \times \left(0.483113194 - 1\right)\frac{0.4 + 2 \times 3.645185154}{\left(0.4 + 3.645185154\right)^2} \\ &\quad -0.483113194 \times \left(0.483113194 - 1\right)\left(2 - 0.483113194\right) \\ &\quad \left(1 - 3 \times 0.483113194\right)\frac{\left(0.4 + 2 \times 3.645185154\right)^2}{2\left(0.4 + 3.645185154\right)^4} \\ &\cong 0.863844672, \end{aligned}$$

and

$$\begin{aligned}
\sigma_h^2 &= 0.483113194^2 \times \frac{2\left(0.4+2\times 3.645185154\right)}{\left(0.4+3.645185154\right)^2} \\
&\quad \left[1-\left(1-0.483113194\right)\left(1-3\times 0.483113194\right)\frac{0.4+2\times 3.645185154}{\left(0.4+3.645185154\right)^2}\right] \\
&\cong 0.243326929.
\end{aligned}$$

Using equation (20), then

$$\begin{aligned}
Q_{\chi^2(0.4,3.645185154)}\left(3.467407377\right) &= \Phi\left[-\frac{\left(\frac{3.467407377}{0.4,+3.645185154}\right)^{0.483113194}-0.863844672}{\sqrt{0.243326929}}\right] \\
&\cong 0.448062178. \qquad (24)
\end{aligned}$$

Finally, combining equations (18), (19) and (24), then

$$\begin{aligned}
p_t &= -5\times e^{-3\%\times 0.5}\times 0.361833+5\times e^{-1\%\times 0.5}\times 0.448062178 \\
&\cong EUR0.44691.
\end{aligned}$$

(b) The terminal payoff of an asset-or-nothing put is equal to

$$ANP_T\left(S,X,T\right)=S_T\mathbb{1}_{\{S_T<X\}}.$$

Therefore,

$$\begin{aligned}
ANP_t\left(S,X,T\right) &= e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left(S_T\mathbb{1}_{\{S_T<X\}}\middle|\mathcal{F}_t\right) \\
&= S_te^{qt}\mathbb{E}_{\mathbb{Q}_S}\left(\frac{S_T\mathbb{1}_{\{S_T<X\}}}{S_Te^{qT}}\middle|\mathcal{F}_t\right) \\
&= S_te^{-q(T-t)}\mathbb{Q}_S\left(S_T<X\middle|\mathcal{F}_t\right). \qquad (25)
\end{aligned}$$

Since

$$\mathbb{Q}_S\left(S_T<X\middle|\mathcal{F}_t\right)=F_{\chi^2\left(2+\frac{2}{2-\beta},2x\right)}\left(2\kappa X^{2-\beta}\right),$$

then equations (19) and (25) yield:

$$\begin{aligned}
ANP_t\left(S,X,T\right) &= 5\times e^{-3\%\times 0.5}\times F_{\chi^2(2.4,3.467407377)}\left(3.645185154\right) \\
&= 5\times e^{-3\%\times 0.5}\times 0.361833 \\
&\cong EUR1.78222989.
\end{aligned}$$

For a contract size of 100 shares, the fair value of the asset-or-nothing put is equal to

$$EUR1.78222989\times 100=EUR178.222989.$$

(a) Using Proposition 22 of the handouts,

$$\begin{aligned} c_0 &= 10 \times e^{-3\% \times 0.25} \times P_1 (S_t = 10, v_t = 0.08; T = 0.25, X = 10) \\ &\quad -e^{-1\% \times 0.25} \times 10 \times P_2 (S_t = 10, v_t = 0.08; T = 0.25, X = 10) . \end{aligned} \tag{26}$$

Using equations (173) and (174) of the handouts:

$$\begin{aligned} P_1 (S_t = 10, v_t = 0.08; T = 0.25, X = 10) &\approx \frac{1}{2} + \frac{7.0593E - 02}{\pi} \\ &\cong 5.2247E - 01, \end{aligned} \tag{27}$$

e

$$\begin{aligned} P_2 (S_t = 10, v_t = 0.08; T = 0.25, X = 10) &\approx \frac{1}{2} + \frac{-0.11005532}{\pi} \\ &\cong 4.6497E - 01. \end{aligned} \tag{28}$$

Combining equations (26), (27) and (28), then

$$\begin{aligned} c_0 &= 10 \times e^{-3\% \times 0.25} \times (5.2247E - 01) - e^{-1\% \times 0.25} \times 10 \times (4.6497E - 01) \\ &\cong EUR0.54759. \end{aligned}$$

(b) The terminal payoff of a range cash-or-nothing option with strikes X_a and X_b, and with a contract size equal do M, is equal to

$$RA_T (S, X_a, X_b, T) = M S_T \mathbb{1}_{\{X_a < S_T < X_b\}}.$$

Consequently,

$$\begin{aligned} RA_t (S, X_a, X_b, T) &= e^{-r(T-t)} M \mathbb{E}_{\mathbb{Q}} \left(S_T \mathbb{1}_{\{X_a < S_T < X_b\}} \middle| \mathcal{F}_t \right) \\ &= S_t e^{qt} M \mathbb{E}_{\mathbb{Q}_S} \left(\frac{S_T \mathbb{1}_{\{X_a < S_T < X_b\}}}{S_T e^{qT}} \middle| \mathcal{F}_t \right) \\ &= M S_t e^{-q(T-t)} \mathbb{Q}_S \left(X_a < S_T < X_b \middle| \mathcal{F}_t \right) \\ &= M S_t e^{-q(T-t)} \left[\mathbb{Q}_S \left(S_T < X_b \middle| \mathcal{F}_t \right) - \mathbb{Q}_S \left(S_T < X_a \middle| \mathcal{F}_t \right) \right] \end{aligned} \tag{29}$$

Since

$$\mathbb{Q}_S \left(S_T < X \middle| \mathcal{F}_t \right) = 1 - P_1 (S_t, v_t; T, X), \tag{30}$$

then, combining equations (29) and (30),

$$RA_t (S, X_a, X_b, T) = M S_t e^{-q(T-t)} \left[P_1 (S_t, v_t; T, X_a) - P_1 (S_t, v_t; T, X_b) \right]. \tag{31}$$

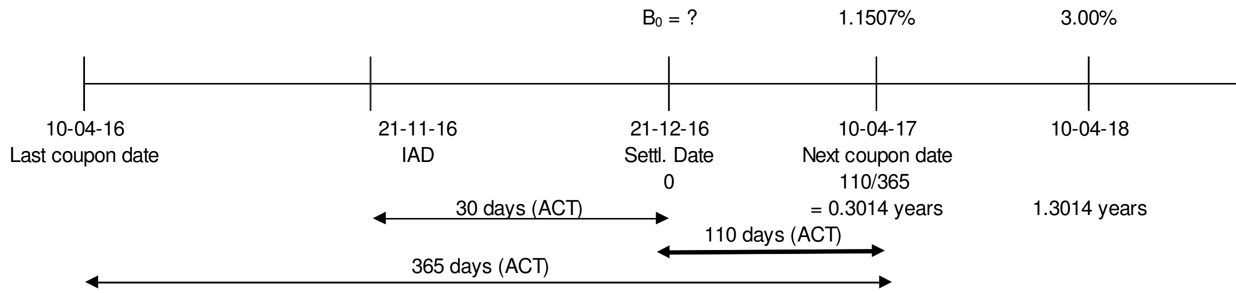
For the option contract under analysis,

$$\begin{aligned} RA_0 &= 100 \times 10 \times e^{-3\% \times 0.25} \times [P_1 (S_t = 10, v_t = 0.08; T = 0.25, X = 10) \\ &\quad -P_1 (S_t = 10, v_t = 0.08; T = 0.25, X = 12)] \\ &= 100 \times 10 \times e^{-3\% \times 0.25} \times \left[(5.2247E - 01) - \left(\frac{1}{2} + \frac{-1.23387061}{\pi} \right) \right] \\ &= 100 \times 10 \times e^{-3\% \times 0.25} \times [(5.2247E - 01) - (1.0725E - 01)] \\ &\cong EUR412.121. \end{aligned}$$

(a) The first coupon is a short coupon and is equal to

$$3\% \times \frac{30 + 110}{365} \cong 1.1507\%.$$

Hence, the purpose is to price a bond with the following future cash flows:



Therefore,

$$\begin{aligned} B_0 &= 1.1507\% P(0, 0.3014) + 3\% P(0, 1.3014) + 103\% P(0, 2.3014) \\ &= 1.1507\% P(0, 0.3014) + 3\% \times 0.9899 + 103\% \times 0.9809. \end{aligned} \tag{32}$$

Concerning the discount factor for the maturity of 0.3014 years, equations (269) and (270) of the handouts imply that

$$\begin{aligned} B(0.3014) &= \frac{1 - e^{-2 \times 0.3014}}{2} \\ &\cong 0.2263, \end{aligned}$$

and

$$\begin{aligned} A(0.3014) &= (0.2263 - 0.3014)\left(1\% - \frac{0.08^2}{2(2)^2}\right) - \frac{0.08^2}{4 \times 2} \times (0.2263)^2 \\ &\cong -0.0007. \end{aligned}$$

Therefore,

$$\begin{aligned} P(0, 0.3014) &= \exp(-0.0007 - 0.2263 \times 0.5\%) \\ &\cong 0.9981. \end{aligned}$$

Recalling equation (32), then

$$\begin{aligned} B_0 &= 1.1507\% \times 0.9981 + 3\% \times 0.9899 + 103\% \times 0.9809 \\ &\cong 105.15\%. \end{aligned}$$

(b) On 21/12/16, the accrued interest amount is equal to

$$AI_0 = 3\% \times \frac{30}{365} \cong 0.247\%.$$

Hence, the gross prices are:

$$GP_0^{bid} = 104.95\% + 0.247\% = 105.197\% > 105.15\%,$$

i.e. we should sell the bond, and

$$GP_0^{bid} = 105.00\% + 0.247\% = 105.247\% > 105.15\%,$$

i.e. we should not buy the bond.

(c) Using Proposition 59 of the handouts,

$$\begin{aligned}
& p_0 \left[P\left(0, 1.3014\right); 99.035\%; 0.3014\right] \\
= & -P\left(0, 1.3014\right) \Phi\left(-d_1^V\right) + 0.99035 \times P\left(0, 0.3014\right) \Phi\left(-d_0^V\right) \\
= & -0.9899 \times \Phi\left(-d_1^V\right) + 0.99035 \times 0.9981 \times \Phi\left(-d_0^V\right),
\end{aligned}$$

onde

$$\begin{aligned}
v\left(0, 0.3014, 1.3014\right) &= \sqrt{\frac{0.08^2}{2^2}\left[1 - e^{-2\times 1}\right]^2 \frac{1 - e^{-2\times 2\times 0.3014}}{2\times 2}} \\
&\cong 1.447\%,
\end{aligned}$$

$$\begin{aligned}
d_1^V &= \frac{\ln\left(\frac{0.9899}{0.99035\times 0.9981}\right) + \frac{(1.447\%)^2}{2}}{1.235\%} \\
&\cong 0.101114134,
\end{aligned}$$

e

$$\begin{aligned}
d_0^V &= 0.101114134 - 1.447\% \\
&\cong 0.086640859.
\end{aligned}$$

Portanto,

$$\begin{aligned}
& p_0 \left[P\left(0, 1.3014\right); 99.035\%; 0.3014\right] \\
= & -0.9899 \times \Phi\left(-0.101114134\right) + 0.99035 \times 0.9981 \times \Phi\left(-0.086640859\right) \\
= & -0.9899 \times 0.459729929 + 0.99035 \times 0.9981 \times 0.465478494 \\
\cong & \ 0.506\%. \qquad (33)
\end{aligned}$$

(d) Using Proposition 61 of the handouts, the fair value of a European-style *put* on a CBB can be decomposed into a portfolio of 2 European-style *puts* on PBDs:

$$\begin{aligned}
& p_0\left(B_0; X = 104.03\%; T = 0.3014\right) \qquad (34) \\
= & \ 3\% \times p_0\left[P\left(0, 1.3014\right); X_1; T = 0.3014\right] + 103\% \times p_0\left[P\left(0, 2.3014\right); X_2; T = 0.3014\right].
\end{aligned}$$

The *strikes* can be obtained through equation (327) of the handouts:

$$\begin{aligned}
X_1 &= \exp\left[A\left(1.3014 - 0.3014\right) - B\left(1\right) \times 1\%\right] \\
&= \exp\left(-0.0054 - 0.4323 \times 1\%\right) \\
&\cong 99.035\%,
\end{aligned}$$

and

$$\begin{aligned}
X_2 &= \exp\left[A\left(2.3014-0.3014\right)-B\left(2\right)\times 1\%\right] \\
&= \exp\left(-0.0141-0.4908\times 1\%\right) \\
&\cong 98.119\%.
\end{aligned}$$

Hence,

$$\begin{aligned}
& p_0\left(B_0; X=104.03\%; T=0.3014\right) \\
= \; & 3\% \times p_0\left[P\left(0,1.3014\right); X_1 = 99.035\%; T=0.3014\right] \\
& +103\% \times p_0\left[P\left(0,2.3014\right); X_2 = 98.119\%; T=0.3014\right].
\end{aligned} \tag{35}$$

The first *put* was already priced in the previous question—please see equation (33):

$$p_0\left[P\left(0,1.3014\right); X_1 = 99.035\%; T=0.3014\right] \cong 0.506\%. \tag{36}$$

Concerning the second *put*, the exam provides the following market price:

$$p_0\left[P\left(0,2.3014\right); X_2 = 98.119\%; T=0.3014\right] = 0.569\%. \tag{37}$$

Combining equations (35), (36) and (37),

$$\begin{aligned}
& p_0\left(B_0; X=104.03\%; T=0.3014\right) \\
= \; & 3\% \times 0.506\% + 103\% \times 0.569\% \\
\cong \; & 0.601\%.
\end{aligned}$$

(a) Assuming that we enter the IRS receiving fixed interest (and paying floating interest), since the intial value of the IRS must be zero, and because the present value of a floating rate bond (at the beginning of a coupon-period and when the coupon rate is in line with its rating), then the fixed interest rate k must be such that

$$0 = \left[kP\left(0,1\right)+kP\left(0,2\right)+\left(1+k\right)P\left(0,3\right)\right]-1,$$

i.e.

$$\begin{aligned}
k &= \frac{1-P\left(0,3\right)}{P\left(0,1\right)+P\left(0,2\right)+P\left(0,3\right)} \\
&= \frac{1-0.9729}{0.9922+0.9826+0.9729} \\
&\cong 0.9199\%.
\end{aligned}$$

(b) The purpose is to price the following option contract:

$$c_0\left[P\left(0,2\right); \frac{P\left(0,2\right)}{P\left(0,1\right)}; 1\right] = c_0\left[P\left(0,2\right); \frac{0.9826}{0.9922} \cong 0.9903; 1\right].$$

Using Proposition 68 of the handouts,

$$\begin{aligned} c_0\left[P\left(0,2\right);99.03\%;1\right] &= P\left(0,2\right)\times F_{\chi^2_{\left(\frac{4\times2\times1\%}{0.08^2},\zeta_2\right)}}\left(\frac{r^*}{L_2}\right) \\ &\quad -0.9903\times P\left(0,1\right)\times F_{\chi^2_{(12.5,\zeta_1)}}\left(\frac{r^*}{L_1}\right), \end{aligned} \tag{38}$$

where

$$\begin{aligned} \gamma &= \sqrt{2^2+2\times\left(8\%\right)^2} \\ &\cong 2.0031974, \end{aligned}$$

$$\begin{aligned} \zeta_2 &= \frac{8r_t\gamma^2e^{\gamma(T_1-t)}}{\sigma^2\left[e^{\gamma(T_1-t)}-1\right]\left\{\gamma\left[e^{\gamma(T_1-t)}+1\right]+\left[k-\sigma^2B\left(T_2-T_1\right)\right]\left[e^{\gamma(T_1-t)}-1\right]\right\}} \\ &= \left[8\times0.5\%\times\left(2.0031974\right)^2\times e^{2.0031974\times1}\right]\left\{0.08^2\times\left(e^{2.0031974\times1}-1\right)\right. \\ &\quad \left[2.0031974\times\left(e^{2.0031974\times1}+1\right)\right. \\ &\quad \left.\left.+\left(2-0.08^2\times(-0.4322)\right)\left(e^{2.0031974\times1}-1\right)\right]\right\}^{-1} \\ &\cong 0.976274579, \end{aligned}$$

$$\begin{aligned} L_2 &= \frac{\sigma^2}{2}\frac{e^{\gamma(T_1-t)}-1}{\gamma\left[e^{\gamma(T_1-t)}+1\right]+\left[k-\sigma^2B\left(T_2-T_1\right)\right]\left[e^{\gamma(T_1-t)}-1\right]} \\ &= \frac{\frac{0.08^2}{2}\times\left(e^{2.0031974\times1}-1\right)}{2.0031974\times\left(e^{2.0031974\times1}+1\right)+\left(2-0.08^2\times(-0.4322)\right)\left(e^{2.0031974\times1}-1\right)} \\ &\cong 0.000691037, \end{aligned}$$

and

$$\begin{aligned} r^* &= \frac{\ln\left(K\right)-A\left(T_2-T_1\right)}{B\left(T_2-T_1\right)} \\ &= \frac{\ln\left(0.9903\right)-A\left(2-1\right)}{B\left(1\right)} \\ &= \frac{\ln\left(0.9903\right)-\left(-0.0057\right)}{-0.4322} \\ &\cong 0.932\%. \end{aligned}$$

Therefore,

$$\begin{aligned} &c_0\left[P\left(0,2\right);99.03\%;1\right] \\ &= 0.9826\times F_{\chi^2_{(12.5,0.976274579)}}\left(\frac{0.932\%}{0.000691037}\cong13.48059247\right) \\ &\quad -0.9903\times0.9922\times F_{\chi^2_{(12.5,0.97685803)}}\left(\frac{0.932\%}{0.00069145}\cong13.47254089\right). \end{aligned} \tag{39}$$

Using the table provided in the exam, we can compute the two probabilities contained in the previous equation:

$$F_{\chi^2_{(12.5,0.976274579)}}\left(13.48059247\right) = 0.55299311, \tag{40}$$

and

$$F_{\chi^2_{(12.5,0.97685803)}}\left(13.47254089\right) = 0.552357741. \tag{41}$$

Finally, combining equations (39), (40) and (41), then

$$\begin{aligned} c_0\left[P\left(0,2\right);99.03\%;1\right] &= 0.9826 \times 0.55299311 - 0.9903 \times 0.9922 \times 0.552357741 \\ &\cong 0.00062432. \end{aligned}$$

Referências