Modelos de Estrutura Temporal de Taxas de Juro
Mestrado em Matemática Financeira 16/17
IBS e FCUL

Exame 1ª Época

19/Dez/17 Duração: 3h

Case 1 Please answer only two of the following questions: (2x2V)

a) Under the Vasiček (1977) model, compute the (time t) *fair value* of a future contract with expiry date at time T_f ($\geq t$), and on a zero-coupon bond $P(t,T)$ with maturity at time T ($\geq T_f$). For this purpose, consider that the future price is a $\mathbb{Q}$-martingale.

b) Under a CEV model with $\beta < 2$, compute the following expected value, for $t < T$: $\mathbb{E}_{\mathbb{Q}}\left[(S_T)^{2-\beta} \middle| \mathcal{F}_t \right]$.

c) Under a multi-factor CIR model, compute the moment generating function of the short-term interest rate under the forward measure $\mathbb{Q}_T$ (that takes as numeraire the zero-coupon bond $P(t,T)$ with maturity at time $T \geq t$), i.e. the expectation $\mathbb{E}_{\mathbb{Q}_T}\left[e^{\alpha r(T)} \middle| \mathcal{F}_t \right]$, for $\alpha \in \mathbb{R}$. For this purpose, remember that, under the multi-factor CIR model

$$r(t) = \sum_{j=1}^{n} Y_j(t),$$

and

$$dY_j(t) = k_j(\theta_j - Y_j(t))\,dt + \sigma_j \sqrt{Y_j(t)}\,dW_j^{\mathbb{Q}}(t),$$

for $j = 1, \ldots, n$, while all Brownian motions $\left\{ W_j^{\mathbb{Q}}(t), t \geq 0 \right\}$ are independent.

Case 2 Consider a CEV process given by the following SDE:

$$dS_t = (r - q)\,S_t dt + \delta S_t^{\frac{\beta}{2}} dW_t^{\mathbb{Q}}.$$

Assume that $S = \$10$, $\beta = -4$, $r = 1\%$, $q = 2\%$ and that the (annualized) standard deviation of stock returns is equal to 20%. Please consider also the following table containing cumulative probabilities associated to a noncentral chi-square random variable with $\frac{1}{3}$ degrees of freedom and a noncentrality parameter equal to 11.19465278:

x	11.00000	11.02799	11.19465
F(x)	0.53029	0.53195	0.54175

Please answer the following questions:

a) Price a European-style standard put on the stock S, with strike equal to \$10 and with a time-to-maturity of 0.25 years. (2V)

b) Price an ATM European-style cash-or-nothing put on the stock S, with a time-to-maturity of 0.25 years, and with a contract size of \$1. (1V)

Case 3 Consider the following parameters for the Heston (1993) model:

- Spot price of the EVN stock = EUR10;
- *Dividend yield* for the EVN stock (continuous compounding) = 2% (30/360);
- Risk-free interest rate (continuous compounding) = 1% (30/360);
- Instantaneous variance of the stock returns (v) = 0.04;
- Speed of mean reversion of the volatility (k) = 3;
- Long-term level of the instantaneous variance (θ) = 0.04;
- Volatility of the instantaneous variance (σ) = 5%; and
- Correlation coefficient between the stock price and the instantaneous variance (ρ) = -0.3.

Next table summarizes the implementation of equations (173) and (174) of the handouts for the strikes EUR10 and EUR8, for a maturity of 6 months, and through a Gauss-Laguerre quadrature with 15 nodes:

w_i	ϕ_i	$X = 10$		$X = 8$	
		$f_1(\phi_i)$	$f_2(\phi_i)$	$f_1(\phi_i)$	$f_2(\phi_i)$
2.1823E-01	9.3308E-02	8.1030E-03	-1.9111E-02	2.5302E-01	2.2582E-01
3.4221E-01	4.9269E-01	1.2057E-02	-2.8398E-02	3.7538E-01	3.3515E-01
2.6303E-01	1.2156E+00	2.4581E-02	-5.7496E-02	7.5352E-01	6.7422E-01
1.2643E-01	2.2699E+00	6.8356E-02	-1.5664E-01	2.0005E+00	1.8017E+00
4.0207E-02	3.6676E+00	2.5734E-01	-5.6334E-01	6.7749E+00	6.1962E+00
8.5639E-03	5.4253E+00	1.2959E+00	-2.6030E+00	2.7667E+01	2.6141E+01
1.2124E-03	7.5659E+00	8.5614E+00	-1.4891E+01	1.2381E+02	1.2535E+02
1.1167E-04	1.0120E+01	7.1947E+01	-1.0016E+02	4.8831E+02	5.9007E+02
6.4599E-06	1.3130E+01	7.3364E+02	-7.3489E+02	2.3672E+02	1.6582E+03
2.2263E-07	1.6654E+01	8.4762E+03	-5.1867E+03	-2.5058E+04	-1.3523E+04
4.2274E-09	2.0776E+01	1.0071E+05	-2.5948E+04	-2.4462E+05	-2.4493E+05
3.9219E-11	2.5624E+01	1.0687E+06	5.2742E+04	1.4695E+05	-8.5332E+05
1.4565E-13	3.1408E+01	8.0487E+06	2.6684E+06	8.9809E+06	7.3535E+06
1.4830E-16	3.8531E+01	2.6529E+07	1.8864E+07	-2.2272E+07	-1.5718E+06
1.6006E-20	4.8026E+01	-5.0677E+05	1.8125E+07	2.3727E+07	4.7617E+06
	$\sum_{i=1}^{15} w_i f_j(\phi_i) =$	0.06795652	-0.12900685	1.34368854	1.26660753

Please answer the following questions:

a) Price a European-style call on the EVN stock, with a strike price equal to EUR8, and with a time-to-maturity of 0.5 years. (1V)

b) Assume that company EVN implements a stock split, and the stock price drops right away to EUR5. Please compute the risk-neutral probability (under measure $\mathbb{Q}$) of the stock price dropping below EUR5 after 6 months. (1V)

Case 4 Consider the following parameters, estimated under measure $\mathbb{Q}$, for the Vašíček (1977) model, and using German treasury bonds for the settlement date of 19/12/17:

alpha	2
gamma	2%
rho	5%
r(t)	1.0%

Next table shows discount factors (for different maturities) based on the previous model' parameters:

T-t	B(t,T)	A(t,T)	P(t,T)
0.5	0.3161	-0.0037	0.9932
1	0.4323	-0.0112	0.9846
1.3068	0.4634	-0.0167	0.9789
2	0.4908	-0.0298	0.9659
2.3068	0.4950	-0.0357	0.9601
3	0.4988	-0.0493	0.9471

Please answer the following questions:

a) Find the fair value of a treasury bond with maturity at 10/04/2020, with *bullet* redemption, and with a coupon rate of 1% (annual coupon under the daycount convention ACT/ACT). For this purpose, consider that the number of calendar days between 10/04/2017 and 19/12/2017 is equal to 253 days. (2V)

b) Formulate a trading decision, knowing that the treasury bond possesses a yield-to-maturity equal to 1.848%(bid)-1.803%(ask). (1V)

c) Price a ATM-forward European-style put, with maturity at 10/04/2018 and on a Treasury Bill with maturity at 10/04/2020. (2V)

Case 5 Consider the following parameters for the Cox, Ingersoll and Ross (1985) model, estimated under measure $\mathbb{Q}$ and using IRS quotes:

k	3.0
theta	2.0%
sigma	10.0%
r	1.0%

Next table contains discount factors (for different maturities) based on the previous model' parameters:

T-t	B(T-t)	A(T-t)	P(t,T)
0.5	-0.25890462	-0.00482035	0.99261799
1	-0.31660832	-0.01366192	0.98331280
1.5	-0.32946384	-0.02339954	0.97365896
2	-0.33232759	-0.03333678	0.96400379
2.5	-0.33296552	-0.04331848	0.95442316
3	-0.33310763	-0.05331009	0.94493308

Consider also the following table containing cumulative probabilities associated to a noncentral chi-square random variable with 24 degrees of freedom and a noncentrality parameter equal to b:

	F(x)		
x	24.2724438	24.3102593	24.3224438
b			
0.62726019	0.51833337	0.52047305	0.52116187
0.62757457	0.51831579	0.52045546	0.52114428

The market also trades European-style options with a time-to-maturity of one year, and on zero-coupon bonds (with money-market credit risk and) with a time-to-maturity of 1.5 years:

	strike:	98.044%	99.025%
Call		0.958%	0.053%
Put		3.56E-10	0.059%

Please answer the following questions:

a) Find the fixed interest rate for a 2-years interest rate swap starting today, with semi-annual compounding (under the 30/360 daycount convention), and quoted against the 6-month Euribor. (1V)

b) Price a European-style call with a time-to-maturity of one year, with a strike equal to 98.044%, and on a zero-coupon bond with a time-to-maturity of two years, knowing that $L_2 = 0.000791124$ and $\zeta_2 = 0.627260186$. (2V)

c) Price a European-style call with a time-to-maturity of one year, with a strike equal to 101%, and on a coupon-bearing bond with a time-to-maturity of two years, a semi-annual coupon, a annual coupon rate of 3% (30/360 daycount convention) and bullet redemption. For this purpose, consider that an instantaneous interest rate of 1.924% yields, after one year, a fair value of 101% for the underlying coupon-bearing bond. (3V)

References

Cox, J., J. Ingersoll, and S. Ross, 1985, A Theory of the Term Structure of Interest Rates, *Econometrica* 53, 385–407.

Heston, S., 1993, A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, *Review of Financial Studies* 6, 327–343.

Vasiček, O., 1977, An Equilibrium Characterization of the Term Structure, *Journal of Financial Economics* 5, 177–188.