Modelos de Estrutura Temporal de Taxas de Juro
Mestrado em Matemática Financeira 18/19
IBS e FCUL

Exame 1ª Época

17/Dez/19 Duração: 3h

Case 1 Please answer only two of the following questions: (2x2V)

a) Show that, under the Vasiček (1977) model, the forward interest rate $E(t, T_1, T_2)$ expected at time t to prevail between times T_1 and T_2 (with a compounding period equal to $T_2 - T_1$ years) is the expectation, under the forward measure that takes as numeraire a zero-coupon bond with maturity at time T_2, for $t \leq T_1 \leq T_2$, of the Euribor rate that will prevail between times T_1 and T_2.

b) Using the Cox, Ingersoll and Ross (1985) model and under the *risk-neutral probability measure* $\mathbb{Q}$, please compute the variance of the random variable $\int_{t_0}^{t} r_s ds$, for $t_0 \leq t$ and conditional to $\mathcal{F}_{t_0}$.

c) Under the Cox et al. (1985) model, compute the (time t) *fair value* of a *cash-or-nothing put* with expiry date at time T, with a *contract size* equal to M, with a *strike* equal to k, and on the Euribor rate $E(T, T+\delta)$ to prevail between times T $(\geq t)$ and $T+\delta$ (with $\delta > 0$). For this purpose, consider that $P(T, T+\delta) = [1 + \delta \times E(T, T+\delta)]^{-1}$.

Case 2 Consider a CEV process given by the following SDE for the stock price GN:

$$dS_t = (r - q) S_t dt + \delta S_t^{\frac{\beta}{2}} dW_t^{\mathbb{Q}}.$$

Assume that $S = \$10$, $\beta = 1$, $r = 1\%$, $q = 2\%$ and that the (annualized) standard deviation of stock returns is equal to 20%. Please consider also the following table containing cumulative probabilities associated to a noncentral chi-square random variable with 2 degrees of freedom and a noncentrality parameter b equal to 400.5002083 or zero:

x	399.50021	400.50021	402.50021
		b = 400.50021	
F(x)	0.48006499	0.49003759	0.50995787
		b = 0	
F(x)	1	1	1

Please answer the following questions:

a) Price a European-style standard put on the stock S, with strike equal to \$10 and with a time-to-maturity of 0.25 years. (2V)

b) How much are you willing to pay today for a contract that promises EUR10 after 3 months if there is bankruptcy of the GN company over the next 3 months? (1V)

Case 3 Consider the following parameters for the Heston (1993) model:

- Spot price of the EVN stock = EUR10;
- *Dividend yield* for the EVN stock (continuous compounding) = 2% (30/360);
- Risk-free interest rate (continuous compounding) = 3% (30/360);
- Instantaneous variance of the stock returns (v) = 0.04;
- Speed of mean reversion of the volatility (k) = 4;
- Long-term level of the instantaneous variance (θ) = 0.06;
- Volatility of the instantaneous variance (σ) = 5%; and
- Correlation coefficient between the stock price and the instantaneous variance (ρ) = -0.6.

Next table summarizes the implementation of equations (220) and (221) of the handouts for the strikes EUR10 and EUR12, for a maturity of 3 months, and through a Gauss-Laguerre quadrature with 15 nodes:

w_i	ϕ_i	$X = 10$		$X = 12$	
		$f_1(\phi_i)$	$f_2(\phi_i)$	$f_1(\phi_i)$	$f_2(\phi_i)$
2.1823E-01	9.3308E-02	9.2256E-03	-3.7538E-03	-1.9091E-01	-2.0389E-01
3.4221E-01	4.9269E-01	1.3741E-02	-5.5827E-03	-2.8389E-01	-3.0314E-01
2.6303E-01	1.2156E+00	2.8172E-02	-1.1353E-02	-5.7702E-01	-6.1557E-01
1.2643E-01	2.2699E+00	7.9659E-02	-3.1335E-02	-1.5902E+00	-1.6917E+00
4.0207E-02	3.6676E+00	3.1156E-01	-1.1593E-01	-5.8677E+00	-6.2019E+00
8.5639E-03	5.4253E+00	1.6917E+00	-5.6322E-01	-2.8447E+01	-2.9675E+01
1.2124E-03	7.5659E+00	1.2806E+01	-3.4566E+00	-1.7605E+02	-1.7907E+02
1.1167E-04	1.0120E+01	1.3571E+02	-2.4361E+01	-1.3238E+03	-1.2806E+03
6.4599E-06	1.3130E+01	2.0202E+03	-1.2642E+02	-1.0856E+04	-9.3058E+03
2.2263E-07	1.6654E+01	4.2312E+04	3.1577E+03	-6.2756E+04	-2.4014E+04
4.2274E-09	2.0776E+01	1.2467E+06	2.7722E+05	1.3238E+06	2.1620E+06
3.9219E-11	2.5624E+01	5.1618E+07	1.9180E+07	9.8385E+07	1.0710E+08
1.4565E-13	3.1408E+01	2.9967E+09	1.5590E+09	4.2860E+09	3.4178E+09
1.4830E-16	3.8531E+01	2.4143E+11	1.6586E+11	1.1294E+11	2.2020E+09
1.6006E-20	4.8026E+01	2.3701E+13	2.4222E+13	-1.0783E+13	-1.9762E+13
	$\sum_{i=1}^{15} w_i f_j(\phi_i) =$	0.11213063	-0.02401091	-1.40646658	-1.43926511

Please answer the following questions:

a) Price a European-style call on the EVN stock, with a strike price equal to EUR12, and with a time-to-maturity of 0.25 years. (1V)

b) Price an ATM European-style cash-or-nothing call on the stock S, with a time-to-maturity of 0.25 years, and with a contract size of $EUR1$. (1V)

Case 4 Consider the following parameters, estimated under measure $\mathbb{Q}$, for the Vasiček (1977) model, and using German treasury bonds for the settlement date of 17/12/2019:

alpha	3
gamma	1%
rho	5%
r(t)	0.5%

Next table shows discount factors (for different maturities; ACT/ACT) based on the previous model parameters:

T-t	B(t,T)	A(t,T)	P(t,T)
0.5	0.2590	-0.0024	0.9963
1	0.3167	-0.0068	0.9917
1.3142	0.3269	-0.0098	0.9887
2	0.3325	-0.0165	0.9820
2.3142	0.3330	-0.0196	0.9790
3	0.3333	-0.0263	0.9724

Please answer the following questions:

a) Find the fair value of a treasury bond with maturity at 10/04/2022, with *bullet* redemption, and with a coupon rate of 0.15% (annual coupon under the daycount convention ACT/ACT). For this purpose, consider that the number of calendar days between 17/12/2019 and 10/04/2020 is equal to 115 days. (2V)

b) Find the ask clean price, knowing that the yield-to-maturity (with annual compounding) of the treasury bond is equal to 0.94%(bid)-0.895%(ask). (1V)

c) Price a European-style call, with strike equal to 99.170%, maturity at 10/04/2020 and on a Treasury Bill with maturity at 10/04/2021. (2V)

d) Price a European-style call with maturity at 10/04/2020, with a strike equal to 98.50%, and on the coupon-bearing bond described in question a). For this purpose, consider that an instantaneous interest rate of 0.4984% yields, on 10/04/2020, a fair value of 98.50% for the underlying coupon-bearing bond, and that the market trades European-style options with a time-to-maturity of 0.3142 years, and on German treasury zero-coupon bonds with a time-to-maturity of 2.3142 years: (3V)

	strike	98.204%	99.170%
Call		0.199%	0.011%
Put		0.295%	1.071%

Case 5 Consider the following parameters for the Cox et al. (1985) model, estimated under measure $\mathbb{Q}$ and using IRS quotes:

k	2.0
theta	3.0%
sigma	10.0%
r	1.0%

Next table contains discount factors (for different maturities) based on the previous model' parameters:

T-t	B(T-t)	A(T-t)	P(t,T)
0.5	-0.31597974	-0.00551746	0.99136028
1	-0.43205740	-0.01702401	0.97888160
1.5	-0.47467036	-0.03072996	0.96514526
2	-0.49030997	-0.04524321	0.95109026
2.5	-0.49604942	-0.06005275	0.93705505
3	-0.49815562	-0.07497102	0.92316012

Consider also the following table containing cumulative probabilities associated to a noncentral chi-square random variable with 24 degrees of freedom and a noncentrality parameter equal to b:

	F(x)	
x	28.7886380	28.8155084
b		
1.24822298	0.71265376	0.71380513
1.24938804	0.71259693	0.71374839

Please answer the following questions:

a) Find a quote for a 2-years IRS with annual revolving and under the 30/360 daycount convention. (1V)

b) Price a European-style put with a time-to-maturity of 1 year, a strike price equal to 97%, and on a zero-coupon bond with a time-to-maturity of two years, knowing that $L_2 = 0.001079136$ and $\zeta_2 = 1.248222984$. (2V)

References

Cox, J., J. Ingersoll, and S. Ross, 1985, A Theory of the Term Structure of Interest Rates, *Econometrica* 53, 385–407.

Heston, S., 1993, A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, *Review of Financial Studies* 6, 327–343.

Vasiček, O., 1977, An Equilibrium Characterization of the Term Structure, *Journal of Financial Economics* 5, 177–188.