Modelos de Estrutura Temporal de Taxas de Juro
Mestrado em Matemática Financeira 13/14
IBS e FCUL
Exame 1ª Época - Resolução

18/Dec/15 Duration: 3h

1. (a) Since a futures contract is a $\mathbb{Q}$-martingale, then

$$F_t = M \times \delta \times \mathbb{E}_{\mathbb{Q}} \left[100\% - E(T, T+\delta) | \mathcal{F}_t \right].$$

On the other hand, since

$$P(T, T+\delta) = \frac{1}{1 + \delta \times E(T, T+\delta)},$$

then

$$\begin{aligned} F_t &= M \times \delta \times \mathbb{E}_{\mathbb{Q}} \left\{ 1 - \frac{1}{\delta} \left[\frac{1}{P(T, T+\delta)} - 1 \right] \middle| \mathcal{F}_t \right\} \\ &= M \times \left\{ 1 + \delta - \mathbb{E}_{\mathbb{Q}} \left[P(T, T+\delta)^{-1} \middle| \mathcal{F}_t \right] \right\}. \end{aligned} \tag{1}$$

Using equation (342) of the handouts, then

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}} \left[P(T, T+\delta)^{-1} \middle| \mathcal{F}_t \right] &= \mathbb{E}_{\mathbb{Q}} \left[\exp\left(-A(\delta) - B(\delta) r_T\right) | \mathcal{F}_t \right] \\ &= \exp\left(-A(\delta)\right) \mathbb{E}_{\mathbb{Q}} \left[\exp\left(-B(\delta) r_T\right) | \mathcal{F}_t \right]. \end{aligned} \tag{2}$$

Using Proposition 66 of the handouts with $\lambda = B(\delta)$, $\mu = 0$, $t = T$, and $t_0 = t$, equation (2) becomes

$$\mathbb{E}_{\mathbb{Q}} \left[P(T, T+\delta)^{-1} \middle| \mathcal{F}_t \right] = \exp\left(-A(\delta)\right) \exp\left[\phi_{B(\delta),0}(T-t) - r_t \psi_{B(\delta),0}(T-t) \right]. \tag{3}$$

Finally, and combining equations (1) and (3), we get

$$F_t = M \times (1+\delta) - M \times \exp\left[-A(\delta) + \phi_{B(\delta),0}(T-t) - r_t \psi_{B(\delta),0}(T-t) \right].$$

(b) The analysis will be illustrated for a call option; for put options the reasoning is similar. Let $c_t(B_t; X; T)$ be the time-t value of a European-style call option on the coupon-bearing bond B_t, with strike X and maturity at time T $(t \geq T)$. The underlying coupon-bearing bond pays N_0 cash flows k_i after the option's expiry date, i.e. at times T_i $(> T)$, for $i = 1, ..., N_0$, and, therefore,

$$B_T = \sum_{i=1}^{N_0} k_i P(T, T_i). \tag{4}$$

We want to show that

$$c_t\left(B_t; X; T\right) = \sum_{i=1}^{N_0} k_i c_t\left[P\left(t, T_i\right); X_i; T\right], \tag{5}$$

where $c_t\left[P\left(t, T_i\right); X_i; T\right]$ is given by Proposition 68 of the handouts,

$$X_i = \exp\left[A\left(T_i - T\right) + B\left(T_i - T\right) r^*\right], \tag{6}$$

and the critical short-term interest rate r^* is the implicit solution of the following equation:

$$\sum_{i=1}^{N_0} k_i \exp\left[A\left(T_i - T\right) + B\left(T_i - T\right) r^*\right] = X. \tag{7}$$

Proof:

Using equations (4) and (7), then

$$\begin{aligned} c_T\left(B_T; X; T\right) &= \left(B_T - X\right)^+ \\ &= \left(\sum_{i=1}^{N_0} k_i P\left(T, T_i\right) - X\right)^+ \\ &= \left(\sum_{i=1}^{N_0} k_i P\left(T, T_i\right) - \sum_{i=1}^{N_0} k_i X_i\right)^+ \\ &= \left[\sum_{i=1}^{N_0} k_i\left(P\left(T, T_i\right) - X_i\right)\right]^+. \end{aligned} \tag{8}$$

The crucial step is to recognize that $P\left(T, T_i\right)$ is a decreasing function of r_T because $B\left(T_i - T\right) < 0$. Since $k > 0$ (to ensure mean-reversion) and $\gamma > 0$ (from equation (346) of the handouts), then $e^{\gamma(T_i - T)} > 1$ and $B\left(T_i - T\right) < 0$. Consequently,

$$\left[\sum_{i=1}^{N_0} k_i\left(P\left(T, T_i\right) - X_i\right)\right]^+ = \sum_{i=1}^{N_0} k_i\left[P\left(T, T_i\right) - X_i\right]^+,$$

and equation (8) becomes

$$c_T\left(B_T; X; T\right) = \sum_{i=1}^{N_0} k_i\left[P\left(T, T_i\right) - X_i\right]^+. \tag{9}$$

Therefore,

$$\begin{aligned}
c_t\left(B_t; X; T\right) &= P\left(t, T\right) \mathbb{E}_{\mathbb{Q}_T}\left[\frac{c_T\left(B_t; X; T\right)}{P\left(T, T\right)} | \mathcal{F}_t\right] \\
&= P\left(t, T\right) \mathbb{E}_{\mathbb{Q}_T}\left\{\sum_{i=1}^{N_0} k_i \left[P\left(T, T_i\right) - X_i\right]^+ | \mathcal{F}_t\right\} \\
&= \sum_{i=1}^{N_0} k_i P\left(t, T\right) \mathbb{E}_{\mathbb{Q}_T}\left\{\frac{\left[P\left(T, T_i\right) - X_i\right]^+}{P\left(T, T\right)} | \mathcal{F}_t\right\} \\
&= \sum_{i=1}^{N_0} k_i c_t\left[P\left(t, T_i\right); X_i; T\right].
\end{aligned}$$

(c) In question 1.c) of the exam of 15/Dez/09 it was shown that, under the Vasiček (1977) model, and for $\lambda, \mu \in \mathbb{R}_+$,

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-\lambda r_t} \exp\left(-\mu \int_{t_0}^{t} r_s ds\right) | \mathcal{F}_{t_0}\right] = \exp\left[\phi_{\lambda,\mu}\left(t - t_0\right) - r_{t_0} \psi_{\lambda,\mu}\left(t - t_0\right)\right], \tag{10}$$

where

$$\psi_{\lambda,\mu}\left(t - t_0\right) := \frac{\left(\alpha\lambda - \mu\right) e^{\alpha\left(t - t_0\right)} + \mu}{\alpha}, \tag{11}$$

and

$$\phi_{\lambda,\mu}\left(t - t_0\right) := \alpha\gamma \int_{t_0}^{t} \frac{\left(\alpha\lambda - \mu\right) e^{\alpha\left(s - t_0\right)} + \mu}{\alpha} ds - \frac{1}{2}\rho^2 \int_{t_0}^{t} \left[\frac{\left(\alpha\lambda - \mu\right) e^{\alpha\left(s - t_0\right)} + \mu}{\alpha}\right]^2 ds. \tag{12}$$

Therefore, the moment generating function (with parameter ω) of the random variable

$$y_t := \int_{t_0}^{t} r_s ds$$

is given by

$$\begin{aligned}
M\left(\omega\right) &= \mathbb{E}_{\mathbb{Q}}\left[\exp\left(\omega y_t\right) | \mathcal{F}_{t_0}\right] \\
&= \exp\left[\phi_{0,-\omega}\left(t - t_0\right) - r_{t_0} \psi_{0,-\omega}\left(t - t_0\right)\right].
\end{aligned} \tag{13}$$

On the other hand, we also know that

$$VAR\left(y_t | \mathcal{F}_{t_0}\right) = \mathbb{E}_{\mathbb{Q}}\left(y_t^2 | \mathcal{F}_{t_0}\right) - \left[\mathbb{E}_{\mathbb{Q}}\left(y_t | \mathcal{F}_{t_0}\right)\right]^2. \tag{14}$$

Through the differentiation of the moment generating function (13) we can obtain the moments contained on the right-hand side of equation (14):

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left(y_t | \mathcal{F}_{t_0}\right) &= \left.\frac{\partial M\left(\omega\right)}{\partial \omega}\right|_{\omega=0} \\
&= \left[\left.\frac{\partial \phi_{0,-\omega}\left(t - t_0\right)}{\partial \omega}\right|_{\omega=0} - r_{t_0} \left.\frac{\partial \psi_{0,-\omega}\left(t - t_0\right)}{\partial \omega}\right|_{\omega=0}\right],
\end{aligned} \tag{15}$$

and

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left(y_t^2|\mathcal{F}_{t_0}\right) &= \left.\frac{\partial^2 M(\omega)}{\partial\omega^2}\right|_{\omega=0} \\ &= \left[\left.\frac{\partial^2\phi_{0,-\omega}(t-t_0)}{\partial\omega^2}\right|_{\omega=0} - r_{t_0}\left.\frac{\partial^2\psi_{0,-\omega}(t-t_0)}{\partial\omega^2}\right|_{\omega=0}\right]. \qquad (16)\end{aligned}$$

2. (a) Using equation (58) of the handouts, the fair value of the European-style call (with $\beta < 2$) is given by:

$$c_t(S,X,T) = S_t e^{-q(T-t)} Q_{\chi^2\left(2+\frac{2}{2-\beta},2x\right)}\left(2\kappa X^{2-\beta}\right) - Xe^{-r(T-t)} F_{\chi^2\left(\frac{2}{2-\beta},2\kappa X^{2-\beta}\right)}(2x), \qquad (17)$$

where

$$\kappa := \frac{2(r-q)}{(2-\beta)\delta^2\left[e^{(2-\beta)(r-q)(T-t)}-1\right]}, \qquad (18)$$

and

$$x := \kappa S_t^{2-\beta} e^{(2-\beta)(r-q)(T-t)}. \qquad (19)$$

Since the (annualized) standard deviation of stock returns is equal to 20% per year, then, and using equation (2) of the handouts,

$$\delta = \frac{20\%}{(10)^{\frac{1-2}{2}}} = 0.632456.$$

Using equations (18) and (19),

$$\kappa = \frac{2(1\% - 2\%)}{(2-1)(0.632456)^2\left[e^{(2-1)(1\%-2\%)\times 0.25}-1\right]} \cong 20.02501042,$$

and

$$x = 20.02501042 \times (10)^{2-1} e^{(2-1)(1\%-2\%)\times 0.25} \cong 199.7501042.$$

Hence, equation (17) yields

$$\begin{aligned} c_t &= 10 \times e^{-2\%\times 0.25} \times Q_{\chi^2\left(2+\frac{2}{2-1},2\times 199.7501042\right)}\left(2\times 20.02501042\times 10^{2-1}\right) \\ &\quad -10\times e^{-1\%\times 0.25} \times F_{\chi^2\left(\frac{2}{2-1},2\times 20.02501042\times 10^{2-1}\right)}(2\times 11.22259259) \\ &= 10\times e^{-2\%\times 0.25}\times Q_{\chi^2(4,399.5002083)}(400.5002083) \qquad (20) \\ &\quad -10\times e^{-1\%\times 0.25}\times F_{\chi^2(2,400.5002083)}(399.5002083).\end{aligned}$$

From the table provided in the exam, we know that

$$F_{\chi^2(2,400.5002083)}(399.5002083) = 0.48006. \qquad (21)$$

The probability $Q_{\chi^2(4,399.5002083)}(400.5002083)$ can be computed using the Sankaran approximation, i.e.

$$\begin{aligned} Q_{\chi^2(a,b)}(z) &= 1 - \mathbb{Q}\left(\chi^2(a,b) < z\right) \\ &= 1 - \mathbb{Q}\left\{\left[\frac{\chi^2(a,b)}{a+b}\right]^h < \left(\frac{z}{a+b}\right)^h\right\} \\ &\approx \Phi\left[-\frac{\left(\frac{z}{a+b}\right)^h - \mu_h}{\sigma_h}\right], \end{aligned} \tag{22}$$

where

$$\mu_h := 1 + h(h-1)\frac{a+2b}{(a+b)^2} - h(h-1)(2-h)(1-3h)\frac{(a+2b)^2}{2(a+b)^4}, \tag{23}$$

$$\sigma_h^2 := h^2\frac{2(a+2b)}{(a+b)^2}\left[1 - (1-h)(1-3h)\frac{a+2b}{(a+b)^2}\right], \tag{24}$$

and

$$h := 1 - \frac{2}{3}(a+b)(a+3b)(a+2b)^{-2}. \tag{25}$$

Since $a = 4$ e $b = 399.5002083$, then

$$\begin{aligned} h &= 1 - \frac{2}{3}(4 + 399.5002083)(4 + 3 \times 399.5002083) \\ &\quad (4 + 2 \times 399.5002083)^{-2} \\ &\cong 0.498343696, \end{aligned}$$

$$\begin{aligned} \mu_h &= 1 + 0.498343696 \times (0.498343696 - 1)\frac{4 + 2 \times 399.5002083}{(4 + 399.5002083)^2} \\ &\quad -0.498343696 \times (0.498343696 - 1)(2 - 0.498343696) \\ &\quad (1 - 3 \times 0.498343696)\frac{(4 + 2 \times 399.5002083)^2}{2(4 + 399.5002083)^4} \\ &\cong 0.998764739, \end{aligned}$$

and

$$\begin{aligned} \sigma_h^2 &= 0.498343696^2 \times \frac{2(4 + 2 \times 399.5002083)}{(4 + 399.5002083)^2} \\ &\quad \left[1 - (1 - 0.498343696)(1 - 3 \times 0.498343696)\frac{4 + 2 \times 399.5002083}{(4 + 399.5002083)^2}\right] \\ &\cong 0.002452719. \end{aligned}$$

Using equation (22), then

$$\begin{aligned} Q_{\chi^2(4,399.5002083)}(400.5002083) &= \Phi\left[-\frac{\left(\frac{400.5002083}{4,+.5002083}\right)^{0.498343696} - 0.998764739}{\sqrt{0.002452719}}\right] \\ &\cong 0.519943498. \end{aligned} \tag{26}$$

Finally, combining equations (20), (21) and (26), then

$$\begin{aligned} c_t &= 10 \times e^{-2\% \times 0.25} \times 0.519943498 - 10 \times e^{-1\% \times 0.25} \times 0.48006 \\ &\cong EUR0.38484. \end{aligned}$$

(b) Using equation (47) of the handouts,

$$\begin{aligned} \int_{400.50021}^{\infty} f_{\chi^2(4,b)}(399.50021)\, db &= 1 - Q_{\chi^2(4-2,400.5002083)}(399.50021) \\ &= F_{\chi^2(2,400.5002083)}(399.50021) \\ &= 0.48006, \end{aligned}$$

where the last line follows from equation (21).

3. (a) Using Proposition 22 of the handouts,

$$\begin{aligned} p_0 = & -15 \times e^{-2\% \times 0.5} \times [1 - P_1(S_t = 15, v_t = 0.09; T = 0.5, X = 10)] \\ & + e^{-1\% \times 0.5} \times 10 \times [1 - P_2(S_t = 15, v_t = 0.09; T = 0.5, X = 10)]. \end{aligned} \tag{27}$$

Using equations (173) and (174) of the handouts:

$$\begin{aligned} P_1(S_t = 15, v_t = 0.09; T = 0.5, X = 10) &\approx \frac{1}{2} + \frac{1.50990948}{\pi} \\ &\cong 9.8062E - 01, \end{aligned} \tag{28}$$

and

$$\begin{aligned} P_2(S_t = 15, v_t = 0.09; T = 0.5, X = 10) &\approx \frac{1}{2} + \frac{1.47210171}{\pi} \\ &\cong 9.6858E - 01. \end{aligned} \tag{29}$$

Combining equations (27), (28) and (29), then

$$\begin{aligned} p_0 &= -15 \times e^{-2\% \times 0.5} \times (1 - 9.8062E - 01) + 10 \times e^{-1\% \times 0.5} \times (1 - 9.6858E - 01) \\ &\cong EUR0.02477. \end{aligned}$$

(b) The terminal payoff of a range cash-or-nothing option with strikes X_a and X_b, and with a contract size equal do M, is equal to

$$RCN_T = M \mathbb{1}_{\{X_a < S_T < X_b\}}.$$

Consequently,

$$\begin{aligned} RCN_t &= e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}\left(M \mathbb{1}_{\{X_a < S_T < X_b\}} \middle| \mathcal{F}_t \right) \\ &= M e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}\left(\mathbb{1}_{\{X_a < S_T < X_b\}} \middle| \mathcal{F}_t \right) \\ &= M e^{-r(T-t)} \mathbb{Q}\left(X_a < S_T < X_b \middle| \mathcal{F}_t \right) \\ &= M e^{-r(T-t)} \left[\mathbb{Q}\left(S_T < X_b \middle| \mathcal{F}_t \right) - \mathbb{Q}\left(S_T < X_a \middle| \mathcal{F}_t \right) \right]. \end{aligned} \tag{30}$$

Since

$$\mathbb{Q}\left(S_T < X \middle| \mathcal{F}_t\right) = 1 - P_2\left(S_t, v_t; T, X\right), \tag{31}$$

and combining equations (30) and (31), then

$$RCN_t = Me^{-r(T-t)}\left[P_2\left(S_t, v_t; T, X_a\right) - P_2\left(S_t, v_t; T, X_b\right)\right]. \tag{32}$$

For the option contract under analysis,

$$P_2\left(S_t, v_t; T, X_a = 10\right) \cong 9.6858E - 01$$

and

$$\begin{aligned} P_2\left(S_t, v_t; T, X_b = 20\right) &\approx \frac{1}{2} + \frac{-1.39799306}{\pi} \\ &\cong 5.5005E - 02. \end{aligned}$$

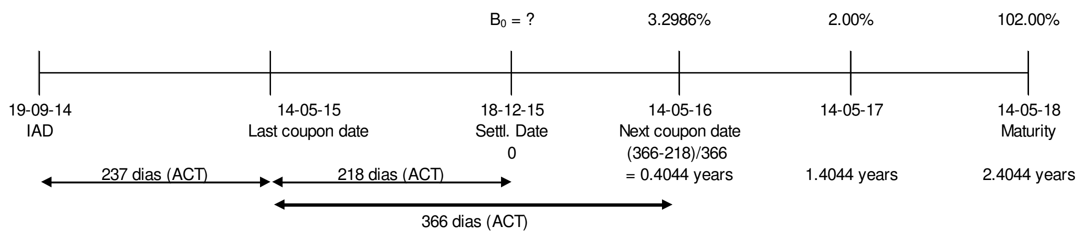
Therefore,

$$\begin{aligned} RCN_0 &= 100 \times e^{-1\% \times 0.5} \times (9.6858E - 01 - 5.5005E - 02) \\ &\cong EUR90.902. \end{aligned}$$

4. (a) The first coupon is a long coupon and is equal to

$$2\% \times \left(1 + \frac{237}{365}\right) \cong 3.2986\%.$$

Hence, the purpose is to price a bond with the following future cash flows:



Therefore,

$$\begin{aligned} B_0 &= 3.2986\%P\left(0, 0.4044\right) + 2\%P\left(0, 1.4044\right) + 102\%P\left(0, 2.4044\right) \\ &= 3.2986\%P\left(0, 0.4044\right) + 2\% \times 0.9866 + 102\% \times 0.9773. \end{aligned} \tag{33}$$

Concerning the discount factor for the maturity of 0.2548 years, equations (269) and (270) of the handouts imply that

$$\begin{aligned} B\left(0.4044\right) &= \frac{1 - e^{-3 \times 0.4044}}{3} \\ &\cong 0.2342, \end{aligned}$$

and

$$
\begin{aligned}
A(0.4044) &= (0.2342 - 0.4044)\left(1\% - \frac{0.1^2}{2(3)^2}\right) - \frac{0.1^2}{4 \times 3} \times (0.2342)^2 \\
&\cong -0.0017.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
P(0, 0.4044) &= \exp(-0.0017 - 0.2342 \times 1\%) \\
&\cong 0.9960.
\end{aligned}
$$

Recalling equation (33), then

$$
\begin{aligned}
B_0 &= 3.2986\% \times 0.9960 + 2\% \times 0.9866 + 102\% \times 0.9773 \\
&\cong 104.941\%.
\end{aligned}
$$

(b) Using Proposition 59 of the handouts,

$$
\begin{aligned}
& c_0 [P(0, 1.4044); 0.9866; 0.4044] \\
= \ & P(0, 1.4044)\,\Phi\left(d_1^V\right) - 0.9866 \times P(0, 0.4044)\,\Phi\left(d_0^V\right) \\
= \ & 0.9866 \times \Phi\left(d_1^V\right) - 0.9866 \times 0.9960 \times \Phi\left(d_0^V\right),
\end{aligned}
$$

where

$$
\begin{aligned}
v(0, 0.4044, 1.4044) &= \sqrt{\frac{0.1^2}{3^2}\left[1 - e^{-3\times 1}\right]^2 \frac{1 - e^{-2\times 3\times 0.4044}}{2 \times 3}} \\
&\cong 1.235\%,
\end{aligned}
$$

$$
\begin{aligned}
d_1^V &= \frac{\ln\left(\frac{0.9866}{0.9866 \times 0.9960}\right) + \frac{(1.235\%)^2}{2}}{1.235\%} \\
&\cong 0.329748238,
\end{aligned}
$$

and

$$
\begin{aligned}
d_0^V &= 0.329748238 - 1.235\% \\
&\cong 0.317402032.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& c_0 [P(0, 1.4044); 0.9866; 0.4044] \\
= \ & 0.9866 \times \Phi(0.329748238) - 0.9866 \times 0.9960 \times \Phi(0.317402032) \\
= \ & 0.9866 \times 0.629204899 - 0.9866 \times 0.9960 \times 0.624530717 \\
\cong \ & 0.707\%. \qquad (34)
\end{aligned}
$$

5. (a) The purpose is to price the following option contract:

$$p_0\left[P\left(0,3\right);94.531\%;1\right].$$

Using Proposition 68 of the handouts,

$$p_0\left[P\left(0,3\right);94.531\%;1\right] = -P\left(0,3\right)\times Q_{\chi^2_{\left(\frac{4\times4\times3\%}{0.05^2},\zeta_2\right)}}\left(\frac{r^*}{L_2}\right) +0.94531\times P\left(0,1\right)\times Q_{\chi^2_{\left(192,\zeta_1\right)}}\left(\frac{r^*}{L_1}\right), \tag{35}$$

where

$$\begin{aligned}\gamma &= \sqrt{4^2+2\times\left(5\%\right)^2}\\ &\cong 4.000624951,\end{aligned}$$

$$\begin{aligned}\zeta_2 &= \frac{8r_t\gamma^2e^{\gamma(T_1-t)}}{\sigma^2\left[e^{\gamma(T_1-t)}-1\right]\left\{\gamma\left[e^{\gamma(T_1-t)}+1\right]+\left[k-\sigma^2B\left(T_2-T_1\right)\right]\left[e^{\gamma(T_1-t)}-1\right]\right\}}\\ &= \left[8\times1\%\times\left(4.000624951\right)^2\times e^{4.000624951\times1}\right]\left\{0.05^2\times\left(e^{4.000624951\times1}-1\right)\right.\\ &\quad\left[4.000624951\times\left(e^{4.000624951\times1}+1\right)\right.\\ &\quad\left.\left.+\left(4-0.05^2\times\left(-0.2499\right)\right)\left(e^{4.000624951\times1}-1\right)\right]\right\}^{-1}\\ &\cong 1.193497622,\end{aligned}$$

$$\begin{aligned}L_2 &= \frac{\sigma^2}{2}\frac{e^{\gamma(T_1-t)}-1}{\gamma\left[e^{\gamma(T_1-t)}+1\right]+\left[k-\sigma^2B\left(T_2-T_1\right)\right]\left[e^{\gamma(T_1-t)}-1\right]}\\ &= \frac{\frac{0.05^2}{2}\times\left(e^{4.000624951\times1}-1\right)}{4.000624951\left(e^{4.000624951\times1}+1\right)+\left(4-0.05^2\left(-0.2499\right)\right)\left(e^{4.000624951\times1}-1\right)}\\ &\cong 0.000153366,\end{aligned}$$

and

$$\begin{aligned}r^* &= \frac{\ln\left(K\right)-A\left(T_2-T_1\right)}{B\left(T_2-T_1\right)}\\ &= \frac{\ln\left(0.94531\right)-A\left(3-1\right)}{B\left(2\right)}\\ &= \frac{\ln\left(0.94531\right)-\left(-0.0525\right)}{-0.2499}\\ &\cong 1.497\%.\end{aligned}$$

Therefore,

$$\begin{aligned}&p_0\left[P\left(0,3\right);94.531\%;1\right]\\ =\ &-0.9185\times Q_{\chi^2_{\left(192,1.193497622\right)}}\left(\frac{1.497\%}{0.000153366}\cong 97.58045157\right)\\ &+0.94531\times0.9752\times Q_{\chi^2_{\left(192,1.193589112\right)}}\left(\frac{1.497\%}{0.00015337}\cong 97.5729719\right).\end{aligned} \tag{36}$$

Using the table provided in the exam, we can compute the two probabilities contained in the previous equation:

$$Q_{\chi^2_{(192,1.193497622)}}(97.58045157) = 1 - 1.33206E - 09, \tag{37}$$

and

$$Q_{\chi^2_{(192,1.193589112)}}(97.5729719) = 1 - 1.32717E - 09. \tag{38}$$

Finally, combining equations (36), (37) and (38), then

$$\begin{aligned} & p_0 \left[P(0,3); 94.531\%; 1\right] \\ = \;& -0.9185 \times (1 - 1.33206E - 09) + 0.94531 \times 0.9752 \times (1 - 1.32717E - 09) \\ \cong \;& 0.00337228. \end{aligned}$$

(b) Using Proposition 61 of the handouts, the fair value of a European-style *put* on a CBB can be decomposed into a portfolio of 2 European-style *puts* on PBDs:

$$\begin{aligned} & p_0 (B_t; X = 98.37\%; T = 1) \\ = \;& 2\% \times p_0 \left[P(0,2); X_1; T = 1\right] + 102\% \times p_0 \left[P(0,3); X_2; T = 1\right]. \end{aligned} \tag{39}$$

The *strikes* can be obtained through equation (327) of the handouts:

$$\begin{aligned} X_1 &= \exp\left[A(2-1) + B(1) \times 1.497\%\right] \\ &= \exp(-0.022636381 - 0.245404429 \times 1.497\%) \\ &\cong 97.403\%, \end{aligned}$$

and

$$\begin{aligned} X_2 &= \exp\left[A(3-1) - B(3-1) \times 1.497\%\right] \\ &= \exp(-0.05249929 - 0.249896711 \times 1.497\%) \\ &\cong 94.531\%. \end{aligned}$$

Hence,

$$\begin{aligned} & p_0 (B_t; X = 98.37\%; T = 1) \\ = \;& 2\% \times p_0 \left[P(0,2); X_1 = 97.403\%; T = 1\right] \\ & +102\% \times p_0 \left[P(0,3); X_2 = 94.531\%; T = 1\right]. \end{aligned} \tag{40}$$

The second *put* was already priced in the previous question—please see equation (34):

$$p_0 \left[P(0,3); 94.531\%; 1\right] \cong 0.00337228. \tag{41}$$

Concerning the first *put*, the exam provides the following market price:

$$p_0 \left[P(0,2); X_1 = 97.403\%; T = 1\right] = 0.341\%. \tag{42}$$

Combining equations (40), (41) and (42),

$$\begin{aligned} & p_0 (B_t; X = 98.37\%; T = 1) \\ = \;& 2\% \times 0.341\% + 102\% \times 0.337228\% \\ \cong \;& 0.351\%. \end{aligned}$$

Referências

Vasiček, O., 1977, An Equilibrium Characterization of the Term Structure, *Journal of Financial Economics* 5, 177–188.