Modelos de Estrutura Temporal de Taxas de Juro
Mestrado em Matemática Financeira 16/17
IBS e FCUL
Exame 1ª Época - Resolução

19/Dec/17 Duration: 3h

1. (a) Since a futures contract is a $\mathbb{Q}$-martingale, then

$$\begin{aligned} F_t &= \mathbb{E}_{\mathbb{Q}}\left[P(T_f,T) \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\frac{P(T_f,T)}{P(T_f,T_f)} \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[P(T_f,T_f,T) \middle| \mathcal{F}_t \right]. \end{aligned} \tag{1}$$

Using equation (315) of the handouts, then

$$\begin{aligned} P(T_f,T_f,T) = {} & \frac{P(t,T)}{P(t,T_f)} \exp\left\{ -\frac{\rho^2}{2} \int_t^{T_f} \left[B^2(T-s) - B^2(T_f-s) \right] ds \right. \\ & \left. -\rho \int_t^{T_f} \left[B(T-s) - B(T_f-s) \right] dW_s^{\mathbb{Q}} \right\}. \end{aligned} \tag{2}$$

Since

$$\rho \int_t^{T_f} \left[B(T-s) - B(T_f-s) \right] dW_s^{\mathbb{Q}} \Bigg| \mathcal{F}_t \overset{\mathbb{Q}}{\sim} N^1\left(0, v^2(t,T_f,T)\right),$$

combining equations (1) and (2), and using equation (318) of the handouts, we get

$$\begin{aligned} F_t &= \frac{P(t,T)}{P(t,T_f)} \exp\left\{ -\frac{\rho^2}{2} \int_t^{T_f} \left[B^2(T-s) - B^2(T_f-s) \right] ds \right\} \\ &\quad \times \mathbb{E}_{\mathbb{Q}}\left[e^{-\rho \int_t^{T_f} \left[B(T-s) - B(T_f-s) \right] dW_s^{\mathbb{Q}}} \middle| \mathcal{F}_t \right] \\ &= \frac{P(t,T)}{P(t,T_f)} \exp\left\{ -\frac{\rho^2}{2} \int_t^{T_f} \left[B^2(T-s) - B^2(T_f-s) \right] ds \right. \\ &\quad \left. + (-1) \times 0 + (-1)^2 \times \frac{v^2(t,T_f,T)}{2} \right\} \\ &= \frac{P(t,T)}{P(t,T_f)} \exp\left\{ -\frac{\rho^2}{2} \int_t^{T_f} \left[B^2(T-s) - B^2(T_f-s) \right] ds \right. \\ &\quad \left. + \frac{\rho^2}{2} \int_t^{T_f} \left[B(T-s) - B(T_f-s) \right]^2 ds \right\} \\ &= \frac{P(t,T)}{P(t,T_f)} \exp\left\{ -\rho^2 \int_t^{T_f} B(T-s)\, B(T_f-s)\, ds \right\}. \end{aligned}$$

(b) Using the identity $y_t = S_t^{2-\beta}$, then

$$\mathbb{E}_{\mathbb{Q}}\left[(S_T)^{2-\beta} \middle| \mathcal{F}_t \right] = \mathbb{E}_{\mathbb{Q}}\left[y_T | \mathcal{F}_t \right], \tag{3}$$

where y_T follows a square-root process. The previous first moment can be obtained from the Laplace transform of the square-root process given in Proposition 4 of the handouts:

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-\lambda y_T} \middle| \mathcal{F}_t \right] = \frac{\exp\left(-\frac{\lambda L f}{1+2\lambda L}\right)}{(1+2\lambda L)^{\frac{2k\theta}{\sigma^2}}}, \tag{4}$$

where

$$L := \frac{\sigma^2\left[1 - e^{-k(T-t)}\right]}{4k}, \tag{5}$$

and

$$f := \frac{4 y_t k}{\sigma^2\left[e^{k(T-t)} - 1\right]}. \tag{6}$$

Since

$$\mathbb{E}_{\mathbb{Q}}\left[y_T | \mathcal{F}_t \right] = -\frac{\partial}{\partial \lambda}\mathbb{E}_{\mathbb{Q}}\left[e^{-\lambda y_T} \middle| \mathcal{F}_t \right]\Big|_{\lambda=0},$$

then equation (4) yields

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[y_T | \mathcal{F}_t \right] \\
=\;& \frac{Lf(1+2\lambda L) - \lambda L f 2L}{(1+2\lambda L)^2}\frac{\exp\left(-\frac{\lambda L f}{1+2\lambda L}\right)}{(1+2\lambda L)^{\frac{2k\theta}{\sigma^2}}} + \frac{2k\theta}{\sigma^2}\frac{\exp\left(-\frac{\lambda L f}{1+2\lambda L}\right)}{(1+2\lambda L)^{\frac{2k\theta}{\sigma^2}+1}} 2L\Bigg|_{\lambda=0} \\
=\;& Lf + \frac{2k\theta}{\sigma^2}2L
\end{aligned} \tag{7}$$

Combining equations (5), (6) and (7), then

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[y_T | \mathcal{F}_t \right] &= \frac{\sigma^2\left[1 - e^{-k(T-t)}\right]}{4k}\frac{4 y_t k}{\sigma^2\left[e^{k(T-t)} - 1\right]} + \frac{4k\theta}{\sigma^2}\frac{\sigma^2\left[1 - e^{-k(T-t)}\right]}{4k} \\
&= e^{-k(T-t)} y_t + \theta\left[1 - e^{-k(T-t)}\right] \\
&= \theta + (y_t - \theta)\, e^{-k(T-t)}.
\end{aligned}$$

(c) Since $r(t) = \sum_{j=1}^{n} Y_j(t)$, then

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_T}\left[e^{\alpha r(T)} \middle| \mathcal{F}_t \right] &= \mathbb{E}_{\mathbb{Q}_T}\left[\exp\left(\alpha \sum_{j=1}^{n} Y_j(T) \right) \middle| \mathcal{F}_t \right] \\
&= \mathbb{E}_{\mathbb{Q}_T}\left[\prod_{j=1}^{n} \exp\left(\alpha Y_j(T)\right) \middle| \mathcal{F}_t \right] \\
&= \prod_{j=1}^{n} \mathbb{E}_{\mathbb{Q}_T}\left[\exp\left(\alpha Y_j(T)\right) | \mathcal{F}_t \right],
\end{aligned} \tag{8}$$

where the last equality arises because the processes $\{Y_j(T)\}_{j=1,\ldots,n}$ are independent.

Using equation (394) of the handouts, then

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}_T}\left[\exp\left(\alpha Y_j(T)\right)\middle|\mathcal{F}_t\right] &= \mathbb{E}_{\mathbb{Q}_T}\left[\exp\left(-\left(-\alpha L_j\right)\frac{Y_j(T)}{L_j}\right)\middle|\mathcal{F}_t\right] \\ &= \frac{\exp\left(-\frac{-\alpha L_j}{1+2\alpha L_j}\zeta_j\right)}{\left(1-2\alpha L_j\right)^{\frac{2k_j\theta_j}{\sigma_j^2}}}, \end{aligned} \tag{9}$$

where

$$L_j = \frac{\sigma_j^2}{2}\frac{e^{\gamma_j(T-t)}-1}{\gamma_j\left[e^{\gamma_j(T-t)}+1\right]+k_j\left[e^{\gamma_j(T-t)}-1\right]},$$

and

$$\zeta_j = \frac{8Y_j(t)\,\gamma_j^2 e^{\gamma_j(T-t)}}{\sigma_j^2\left[e^{\gamma_j(T-t)}-1\right]\left\{\gamma_j\left[e^{\gamma_j(T-t)}+1\right]+k_j\left[e^{\gamma_j(T-t)}-1\right]\right\}},$$

with $\gamma_j = \sqrt{k_j^2+2\sigma_j^2}$.

Finally, combining equations (8) and (9), then

$$\mathbb{E}_{\mathbb{Q}_T}\left[e^{\alpha r(T)}\middle|\mathcal{F}_t\right] = \prod_{j=1}^{n}\frac{\exp\left(-\frac{-\alpha L_j}{1+2\alpha L_j}\zeta_j\right)}{\left(1-2\alpha L_j\right)^{\frac{2k_j\theta_j}{\sigma_j^2}}}.$$

(a) Using equation (59) of the handouts, the fair value of the European-style put (with $\beta < 2$) is given by:

$$p_t(S,X,T) = -S_t e^{-q(T-t)}F_{\chi^2\left(2+\frac{2}{2-\beta},2x\right)}\left(2\kappa X^{2-\beta}\right)+Xe^{-r(T-t)}Q_{\chi^2\left(\frac{2}{2-\beta},2\kappa X^{2-\beta}\right)}\left(2x\right), \tag{10}$$

where

$$\kappa := \frac{2(r-q)}{(2-\beta)\,\delta^2\left[e^{(2-\beta)(r-q)(T-t)}-1\right]}, \tag{11}$$

and

$$x := \kappa S_t^{2-\beta}e^{(2-\beta)(r-q)(T-t)}. \tag{12}$$

Since the (annualized) standard deviation of stock returns is equal to 20% per year, then, and using equation (2) of the handouts,

$$\delta = \frac{20\%}{(10)^{\frac{-4-2}{2}}} = 200.$$

Using equations (11) and (12),

$$\kappa = \frac{2\,(1\%-2\%)}{(2+4)\,(200)^2\left[e^{(2+4)(1\%-2\%)\times 0.25}-1\right]} \cong 5.59733E-06,$$

and

$$x=(5.59733E-06)\times(10)^{2+4}e^{(2+4)(1\%-2\%)\times 0.25}\cong 5.513993055.$$

Hence, equation (10) yields

$$\begin{aligned}
p_t &= -10\times e^{-2\%\times 0.25}\times F_{\chi^2\left(2+\frac{2}{2+4},2\times 5.513993055\right)}\left(2\times(5.59733E-06)\times 10^{2+4}\right)\\
&\quad +10\times e^{-1\%\times 0.25}\times Q_{\chi^2\left(\frac{2}{2+4},2\times(5.59733E-06)\times 10^{2+4}\right)}(2\times 5.513993055)\\
&= -10\times e^{-2\%\times 0.25}\times F_{\chi^2(2.3333,11.02798611)}(11.19465278) \qquad (13)\\
&\quad +10\times e^{-1\%\times 0.25}\times Q_{\chi^2(0.3333,11.19465278)}(11.02798611).
\end{aligned}$$

From the table provided in the exam, we know that

$$Q_{\chi^2(0.3333,11.19465278)}(11.02798611)=1-0.53195=0.468050884. \qquad (14)$$

The probability $F_{\chi^2(2.3333,11.02798611)}(11.19465278)$ can be computed using the Sankaran approximation, i.e.

$$\begin{aligned}
F_{\chi^2(a,b)}(z) &= \mathbb{Q}\left(\chi^2(a,b)<z\right)\\
&= \mathbb{Q}\left\{\left[\frac{\chi^2(a,b)}{a+b}\right]^h<\left(\frac{z}{a+b}\right)^h\right\}\\
&\approx \Phi\left[\frac{\left(\frac{z}{a+b}\right)^h-\mu_h}{\sigma_h}\right], \qquad (15)
\end{aligned}$$

where

$$\mu_h:=1+h(h-1)\frac{a+2b}{(a+b)^2}-h(h-1)(2-h)(1-3h)\frac{(a+2b)^2}{2(a+b)^4}, \qquad (16)$$

$$\sigma_h^2:=h^2\frac{2(a+2b)}{(a+b)^2}\left[1-(1-h)(1-3h)\frac{a+2b}{(a+b)^2}\right], \qquad (17)$$

and

$$h:=1-\frac{2}{3}(a+b)(a+3b)(a+2b)^{-2}. \qquad (18)$$

Since $a=2.3333$ and $b=11.02798611$, then

$$\begin{aligned}
h &= 1-\frac{2}{3}(2.3333+11.02798611)(2.3333+3\times 11.02798611)\\
&\quad (2.3333+2\times 11.02798611)^{-2}\\
&\cong 0.469635353,
\end{aligned}$$

$$\begin{aligned}
\mu_h &= 1+0.469635353\times(0.469635353-1)\frac{2.3333+2\times 11.02798611}{(2.3333+11.02798611)^2}\\
&\quad -0.469635353\times(0.469635353-1)(2-0.469635353)\\
&\quad (1-3\times 0.469635353)\frac{(2.3333+2\times 11.02798611)^2}{2(2.3333+11.02798611)^4}\\
&\cong 0.964517487,
\end{aligned}$$

and

$$\begin{aligned}\sigma_h^2 &= 0.469635353^2 \times \frac{2\,(2.3333 + 2 \times 11.02798611)}{(2.3333 + 11.02798611)^2} \\ &\quad \left[1 - (1 - 0.469635353)\,(1 - 3 \times 0.469635353)\,\frac{2.3333 + 2 \times 11.02798611}{(2.3333 + 11.02798611)^2}\right] \\ &\cong 0.06204868.\end{aligned}$$

Using equation (15), then

$$\begin{aligned}&F_{\chi^2(2.3333,11.02798611)}\,(11.19465278) \\ &= \Phi\left[\frac{\left(\frac{11.19465278}{2.3333+11.02798611}\right)^{0.469635353} - 0.964517487}{\sqrt{0.06204868}}\right] \\ &\cong 0.42950099. \qquad (19)\end{aligned}$$

Finally, combining equations (13), (14) and (19), then

$$\begin{aligned}p_t &= -10 \times e^{-2\%\times 0.25} \times 0.42950099 + 10 \times e^{-1\%\times 0.25} \times 0.468050884 \\ &\cong EUR0.39523.\end{aligned}$$

(b) The terminal payoff of a cash-or-nothing put is equal to

$$CNP_T\,(S, X, T) = \mathbb{1}_{\{S_T < X\}}.$$

Therefore,

$$\begin{aligned}CNP_t\,(S, X, T) &= e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left(\mathbb{1}_{\{S_T<X\}}\middle|\,\mathcal{F}_t\right) \\ &= e^{-r(T-t)}\mathbb{Q}\,(S_T < X|\,\mathcal{F}_t)\,. \qquad (20)\end{aligned}$$

Since

$$\mathbb{Q}\,(S_T < X|\,\mathcal{F}_t) = Q_{\chi^2\left(\frac{2}{2-\beta},2\kappa X^{2-\beta}\right)}\,(2x)\,,$$

then equations (14) and (20) yield:

$$\begin{aligned}CNP_t\,(S, X, T) &= e^{-1\%\times 0.25} \times Q_{\chi^2\left(\frac{2}{2+4},2\times(5.59733E-06)\times 10^{2+4}\right)}\,(2 \times 5.513993055) \\ &= e^{-1\%\times 0.25} \times 0.468050884 \\ &\cong \$0.466882218.\end{aligned}$$

(a) Using Proposition 22 of the handouts,

$$\begin{aligned}c_0 &= 10 \times e^{-2\%\times 0.5} \times P_1\,(S_t = 10, v_t = 0.04; T = 0.5, X = 8) \qquad (21) \\ &\quad -e^{-1\%\times 0.5} \times 8 \times [S_t = 10, v_t = 0.04; T = 0.5, X = 8]\,.\end{aligned}$$

Using equations (173) and (174) of the handouts:

$$\begin{aligned}P_1\,(S_t = 10, v_t = 0.04; T = 0.5, X = 8) &\approx \frac{1}{2} + \frac{1.34368854}{\pi} \\ &\cong 0.92771, \qquad (22)\end{aligned}$$

and

$$
\begin{aligned}
P_2\left(S_t = 10, v_t = 0.04; T = 0.5, X = 8\right) &\approx \frac{1}{2} + \frac{1.26660753}{\pi} \\
&\cong 0.90317. \qquad (23)
\end{aligned}
$$

Combining equations (21), (22) and (23), then

$$
\begin{aligned}
c_0 &= 10 \times e^{-2\% \times 0.5} \times 0.92771 - e^{-1\% \times 0.5} \times 8 \times 0.90317 \\
&\cong EUR1.99543.
\end{aligned}
$$

(b) The purpose is to compute the following probability:

$$
\mathbb{Q}\left(S_T < 5 \middle| \mathcal{F}_t\right) = 1 - P_2\left(S_t = 5, v_t = 0.04; T = 0.5, X = 5\right). \qquad (24)
$$

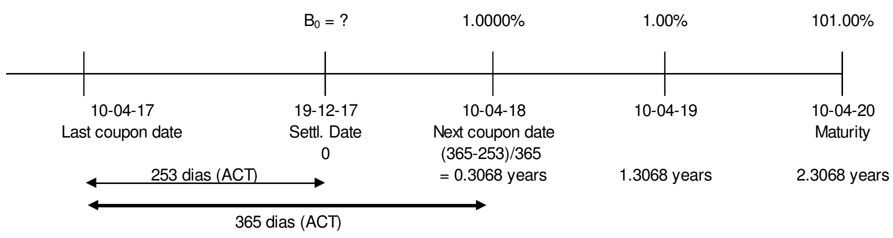
However, $P_j\left(S_t, v_t; T, X\right)$ are homogeneous functions of degree zero in the spot and the strike because

$$
\begin{aligned}
& P_j\left(\lambda S_t, v_t; T, \lambda X\right) \\
&= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{\exp\left(-i\phi \ln\left(\lambda X\right)\right)\varphi_j\left(\lambda S_t, v_t; T, \phi\right)}{i\phi}\right] d\phi \\
&= \frac{1}{2} \\
&\quad + \frac{\int_0^\infty \mathrm{Re}\left\{\frac{\exp(-i\phi\ln(\lambda X))\exp[C_j(\phi;T-t)+D_j(\phi;T-t)v_t+i\phi\ln(\lambda S_t)]}{i\phi}\right\} d\phi}{\pi} \\
&= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{\exp\left(-i\phi \ln\left(X\right)\right)\varphi_j\left(S_t, v_t; T, \phi\right)}{i\phi}\right] d\phi \\
&= P_j\left(S_t, v_t; T, X\right). \qquad (25)
\end{aligned}
$$

Combining equations (24) and (25),

$$
\begin{aligned}
\mathbb{Q}\left(S_T < 5 \middle| \mathcal{F}_t\right) &= 1 - P_2\left(S_t = 5, v_t = 0.04; T = 0.5, X = 5\right) \\
&= 1 - P_2\left(S_t = 10, v_t = 0.04; T = 0.5, X = 10\right) \\
&= 1 - \left(\frac{1}{2} + \frac{-0.12900685}{\pi}\right) \\
&= 1 - 0.45894 \\
&= 0.54106.
\end{aligned}
$$

(a) The purpose is to price a bond with the following future cash flows:



Therefore,

$$
\begin{aligned}
B_0 &= 1\%P\left(0,0.3068\right)+1\%P\left(0,1.3068\right)+101\%P\left(0,2.3068\right)\\
&= 1\%P\left(0,0.3068\right)+1\%\times 0.9789+101\%\times 0.9601. \qquad (26)
\end{aligned}
$$

Concerning the discount factor for the maturity of 0.3068 years, equations (269) and (270) of the handouts imply that

$$
\begin{aligned}
B\left(0.3068\right) &= \frac{1-e^{-2\times 0.3068}}{2}\\
&\cong 0.2293,
\end{aligned}
$$

and

$$
\begin{aligned}
A\left(0.3068\right) &= \left(0.2293-0.3068\right)\left(2\%-\frac{0.05^2}{2\left(2\right)^2}\right)-\frac{0.05^2}{4\times 2}\times\left(0.2293\right)^2\\
&\cong -0.0015.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
P\left(0,0.3068\right) &= \exp\left(-0.0015-0.2293\times 1\%\right)\\
&\cong 0.9962.
\end{aligned}
$$

Recalling equation (26), then

$$
\begin{aligned}
B_0 &= 1\%\times 0.9962+1\%\times 0.9789+101\%\times 0.9601\\
&\cong 98.947\%.
\end{aligned}
$$

(b) Since the YTM is the (flat) discount rate that turns the present value of all future cash flows equal to the bond's gross price, then

$$
GP_0^{bid}=\frac{1\%}{\left(1+1.848\%\right)^{0.3068}}+\frac{1\%}{\left(1+1.848\%\right)^{1.3068}}+\frac{101\%}{\left(1+1.848\%\right)^{2.3068}}\cong 98.793\%<B_0,
$$

i.e. we shall not sell, and

$$
GP_0^{ask}=\frac{1\%}{\left(1+1.803\%\right)^{0.3068}}+\frac{1\%}{\left(1+1.803\%\right)^{1.3068}}+\frac{101\%}{\left(1+1.803\%\right)^{2.3068}}\cong 98.893\%<B_0,
$$

i.e. we shall buy the bond.

(c) Using Proposition 59 of the handouts,

$$
\begin{aligned}
& p_0\left[P\left(0,2.3068\right);K=\frac{P\left(0,2.3068\right)}{P\left(0,0.3068\right)}=\frac{0.9601}{0.9962}\cong 0.9638;0.3068\right]\\
&= -P\left(0,2.3068\right)\times\Phi\left(-d_1^V\right)+0.9638\times P\left(0,0.3068\right)\times\Phi\left(-d_0^V\right)\\
&= -0.9601\times\Phi\left(-d_1^V\right)+0.9638\times 0.9962\times\Phi\left(-d_0^V\right),
\end{aligned}
$$

where

$$
\begin{aligned}
v(0, 0.3068, 2.3068) &= \sqrt{\frac{0.05^2}{2^2}\left[1-e^{-2\times 2}\right]^2 \frac{1-e^{-2\times 2\times 0.3068}}{2\times 2}} \\
&\cong 1.032\%,
\end{aligned}
$$

$$
\begin{aligned}
d_1^V &= \frac{\ln\left(\frac{0.9601}{0.9638\times 0.9962}\right)+\frac{(1.032\%)^2}{2}}{1.032\%} \\
&\cong 0.005158755,
\end{aligned}
$$

and

$$
\begin{aligned}
d_0^V &= 0.005158755 - 1.032\% \\
&\cong -0.005158755.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& p_0\left[P(0, 2.3068); K = 0.9638; 0.3068\right] \\
=\ & -0.9601 \times \Phi(-0.005158755) + 0.9638 \times 0.9962 \times \Phi(0.005158755) \\
=\ & -0.9601 \times 0.497941964 + 0.9638 \times 0.9962 \times 0.502058036 \\
\cong\ & 0.395\%. \qquad (27)
\end{aligned}
$$

(a) Assuming that we enter the IRS receiving fixed interest (and paying floating interest), since the initial value of the IRS must be zero, and because the present value of a floating rate bond (at the beginning of a coupon-period and when the coupon rate is in line with its rating), then the fixed interest rate k must be such that

$$
0 = \left[\frac{k}{2}P(0, 0.5) + \frac{k}{2}P(0, 1) + \frac{k}{2}P(0, 1.5) + \left(1+\frac{k}{2}\right)P(0, 2)\right] - 1,
$$

i.e.

$$
\begin{aligned}
k &= 2 \times \frac{1 - P(0,2)}{P(0,0.5) + P(0,1) + P(0,1.5) + P(0,2)} \\
&= 2 \times \frac{1 - 0.96400379}{0.99261799 + 0.98331280 + 0.97365896 + 0.96400379} \\
&\cong 7.468\%.
\end{aligned}
$$

(b) The purpose is to price the following option contract:

$$
c_0\left[P(0, 2); 98.044\%; 1\right].
$$

Using Proposition 68 of the handouts,

$$
\begin{aligned}
c_0\left[P(0, 2); 98.044\%; 1\right] &= P(0, 2) \times F_{\chi^2_{\left(\frac{4\times 3\times 2\%}{0.1^2}, \zeta_2\right)}}\left(\frac{r^*}{L_2}\right) \qquad (28) \\
&\quad -0.98044 \times P(0, 1) \times F_{\chi^2_{(24, \zeta_1)}}\left(\frac{r^*}{L_1}\right),
\end{aligned}
$$

where

$$\begin{aligned}\gamma &= \sqrt{3^2 + 2 \times (10\%)^2} \\ &\cong 3.003331484,\end{aligned}$$

$$\begin{aligned}\zeta_1 &= \frac{8 r_t \gamma^2 e^{\gamma(T_1 - t)}}{\sigma^2 \left[e^{\gamma(T_1-t)} - 1\right] \left\{\gamma \left[e^{\gamma(T_1-t)} + 1\right] + k \left[e^{\gamma(T_1-t)} - 1\right]\right\}} \\ &= \left[8 \times 1\% \times (3.003331484)^2 \times e^{3.003331484 \times 1}\right] \left\{0.1^2 \times \left(e^{3.003331484\times 1} - 1\right)\right. \\ &\quad \left[3.003331484 \times \left(e^{3.003331484 \times 1} + 1\right)\right. \\ &\quad \left.\left. +3 \times \left(e^{3.003331484\times 1} - 1\right)\right]\right\}^{-1} \\ &\cong 0.627574572,\end{aligned}$$

$$\begin{aligned}L_1 &= \frac{\sigma^2}{2} \frac{e^{\gamma(T_1-t)} - 1}{\gamma \left[e^{\gamma(T_1-t)} + 1\right] + k \left[e^{\gamma(T_1-t)} - 1\right]} \\ &= \frac{\frac{0.1^2}{2} \times \left(e^{3.003331484\times 1} - 1\right)}{3.003331484 \times \left(e^{3.003331484\times 1} + 1\right) + 3 \times \left(e^{3.003331484 \times 1} - 1\right)} \\ &\cong 0.000791521,\end{aligned}$$

and

$$\begin{aligned}r^* &= \frac{\ln(K) - A(T_2 - T_1)}{B(T_2 - T_1)} \\ &= \frac{\ln(0.98044) - A(2-1)}{B(1)} \\ &= \frac{\ln(0.98044) - (-0.01366192)}{-0.31660832} \\ &\cong 1.924\%.\end{aligned}$$

Therefore,

$$\begin{aligned}&c_0 \left[P(0,2); 98.044\%; 1\right] \qquad (29)\\ &= 0.96400379 \times F_{\chi^2_{(24, 0.627260186)}} \left(\frac{1.924\%}{0.000791124} \cong 24.32244375\right) \\ &\quad -0.98044 \times 0.98331280 \times F_{\chi^2_{(24,0.627574572)}} \left(\frac{1.924\%}{0.000791521} \cong 24.31025935\right).\end{aligned}$$

Using the table provided in the exam, we can compute the two probabilities contained in the previous equation:

$$F_{\chi^2_{(24,0.627260186)}} (24.32244375) = 0.52116187, \qquad (30)$$

and

$$F_{\chi^2_{(24,0.627574572)}} (24.31025935) = 0.52045546. \qquad (31)$$

Finally, combining equations (29), (30) and (31), then

$$
\begin{aligned}
& c_0\left[P(0,2);98.044\%;1\right] \\
= \; & 0.96400379 \times 0.52116187 - 0.98044 \times 0.98331280 \times 0.52045546 \\
\cong \; & 0.00064188.
\end{aligned}
$$

(c) Using Proposition 61 of the handouts, the fair value of a European-style *call* on a CBB can be decomposed into a portfolio of 2 European-style *calls* on PBDs:

$$
\begin{aligned}
& c_0\left(B_t;X=101\%;T=1\right) \\
= \; & 1.5\% \times c_0\left[P(0,1.5);X_1;T=1\right] + 101.5\% \times c_0\left[P(0,2);X_2;T=1\right].
\end{aligned} \tag{32}
$$

The *strikes* can be obtained through equation (327) of the handouts:

$$
\begin{aligned}
X_1 &= \exp\left[A(1.5-1) + B(0.5) \times 1.924\%\right] \\
&= \exp\left(-0.00482035 - 0.25890462 \times 1.924\%\right) \\
&\cong 99.025\%,
\end{aligned}
$$

and

$$
\begin{aligned}
X_2 &= \exp\left[A(2-1) - B(2-1) \times 1.924\%\right] \\
&= \exp\left(-0.01366192 - 0.31660832 \times 1.924\%\right) \\
&\cong 98.044\%.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& c_0\left(B_t;X=101\%;T=1\right) \\
= \; & 1.5\% \times c_0\left[P(0,1.5);X_1=99.025\%;T=1\right] \\
& +101.5\% \times c_0\left[P(0,2);X_2=98.044\%;T=1\right].
\end{aligned} \tag{33}
$$

The second *call* was already priced in the previous question—please see equation (27):

$$
c_0\left[P(0,2);X_2=98.044\%;T=1\right] \cong 0.00064188. \tag{34}
$$

Concerning the first *call*, the exam provides the following market price:

$$
c_0\left[P(0,1.5);X_1=99.025\%;T=1\right] = 0.053\%. \tag{35}
$$

Combining equations (33), (34) and (35),

$$
\begin{aligned}
& c_0\left(B_t;X=101\%;T=1\right) \\
= \; & 1.5\% \times 0.053\% + 101.5\% \times 0.064188\% \\
\cong \; & 0.066\%.
\end{aligned}
$$