Modelos de Estrutura Temporal de Taxas de Juro
Mestrado em Matemática Financeira 13/14
IBS e FCUL
Exame 1ª Época - Resolução

22/Dec/14 Duration: 3h

1. (a) Under the Heston (1993) model, the variance of the rate of return on the underlying asset follows a square-root process:

$$dv_t = k(\theta - v_t)\,dt + \sigma\sqrt{v_t}\,dW_2^{\mathbb{Q}}(t).$$

Hence, and using Itô's lemma,

$$\begin{aligned} d\sqrt{v_t} &= \frac{1}{2\sqrt{v_t}}dv_t + \frac{1}{2}\frac{1}{2}\left(-\frac{1}{2}\right)v_t^{-\frac{3}{2}}d\langle v_t, v_t\rangle \\ &= \frac{k}{2}\left(\frac{\theta}{\sqrt{v_t}} - \sqrt{v_t}\right)dt + \frac{\sigma}{2}dW_2^{\mathbb{Q}}(t) - \frac{1}{8}v_t^{-\frac{3}{2}}\sigma^2 v_t dt \\ &= \frac{k}{2}\left(\frac{\theta}{\sqrt{v_t}} - \frac{\sigma^2}{4k\sqrt{v_t}} - \sqrt{v_t}\right)dt + \frac{\sigma}{2}dW_2^{\mathbb{Q}}(t). \end{aligned}$$

Therefore, the standard deviation of the rate of return on the underlying asset follows a *Ornstein-Uhlenbeck* process iff $\theta = \frac{\sigma^2}{4k}$. In this case,

$$d\sqrt{v_t} = -\frac{k}{2}\sqrt{v_t}dt + \frac{\sigma}{2}dW_2^{\mathbb{Q}}(t).$$

However, the Feller condition requires that $\frac{2k\theta}{\sigma^2} > 1$, i.e. $\theta > \frac{\sigma^2}{2k} > \frac{\sigma^2}{4k}$. Consequentely, for $\sqrt{v_t}$ to follow a *Ornstein-Uhlenbeck* process, it would not be possible to satisfy the Feller condition.

(b) For $x_t := \ln(S_t)$, and using equation (132) of the handouts,

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}\left[(S_T)^\alpha \middle| \mathcal{F}_t\right] &= \mathbb{E}_{\mathbb{Q}}\left[\left(e^{\ln(S_T)}\right)^\alpha \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[e^{\alpha x_T} \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[e^{i(-i\alpha)x_T} \middle| \mathcal{F}_t\right] \\ &= \varphi_2(S_t, v_t; T, -i\alpha). \end{aligned}$$

Finally, and using equation (132) of the handouts,

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}\left[(S_T)^\alpha \middle| \mathcal{F}_t\right] &= \exp\left[C_2(-i\alpha; T-t) + D_2(-i\alpha; T-t)v_t + i(-i\alpha)x_t\right] \\ &= \exp\left[C_2(-i\alpha; T-t) + D_2(-i\alpha; T-t)v_t + \alpha x_t\right]. \end{aligned}$$

(c) Using the forward measure $\mathbb{Q}_{T+\delta}$ associated to the numeraire $P(t, T+\delta)$, the time-T value of the call option is

$$\begin{aligned}
& \delta c_T [E(T, T+\delta); k; T+\delta] \\
= \;& \delta P(T, T+\delta) \mathbb{E}_{\mathbb{Q}_{T+\delta}} \left\{ \frac{[E(T, T+\delta) - k]^+}{P(T+\delta, T+\delta)} | \mathcal{F}_T \right\} \\
= \;& P(T, T+\delta) \mathbb{E}_{\mathbb{Q}_{T+\delta}} \left\{ [\delta E(T, T+\delta) - \delta k]^+ | \mathcal{F}_T \right\}. \qquad (1)
\end{aligned}$$

Using the identity

$$P(T, T+\delta) = [1 + \delta \times E(T, T+\delta)]^{-1},$$

i.e.

$$\delta E(T, T+\delta) = P(T, T+\delta)^{-1} - 1,$$

equation (1) can be rewritten as

$$\begin{aligned}
& \delta c_T [E(T, T+\delta); k; T+\delta] \\
= \;& P(T, T+\delta) \mathbb{E}_{\mathbb{Q}_{T+\delta}} \left\{ \left[P(T, T+\delta)^{-1} - (1+\delta k) \right]^+ | \mathcal{F}_T \right\} \\
= \;& \mathbb{E}_{\mathbb{Q}_{T+\delta}} \left\{ [1 - (1+\delta k) P(T, T+\delta)]^+ | \mathcal{F}_T \right\} \\
= \;& (1+\delta k) \mathbb{E}_{\mathbb{Q}_{T+\delta}} \left\{ \left[(1+\delta k)^{-1} - P(T, T+\delta) \right]^+ | \mathcal{F}_T \right\}. \qquad (2)
\end{aligned}$$

Using again the forward measure $\mathbb{Q}_{T+\delta}$ associated to the numeraire $P(t, T+\delta)$ and equation (2), the time-t value of the call option is

$$\begin{aligned}
& \delta c_t [E(T, T+\delta); k; T+\delta] \\
= \;& P(t, T+\delta) \mathbb{E}_{\mathbb{Q}_{T+\delta}} \left\{ \frac{\delta c_T [E(T, T+\delta); k; T+\delta]}{P(T, T+\delta)} | \mathcal{F}_t \right\} \\
= \;& P(t, T+\delta) \mathbb{E}_{\mathbb{Q}_{T+\delta}} \left\{ \frac{(1+\delta k) \mathbb{E}_{\mathbb{Q}_{T+\delta}} \left[\left[(1+\delta k)^{-1} - P(T, T+\delta) \right]^+ | \mathcal{F}_T \right]}{P(T, T+\delta)} | \mathcal{F}_t \right\} \\
= \;& (1+\delta k) P(t, T+\delta) \mathbb{E}_{\mathbb{Q}_{T+\delta}} \left\{ \frac{\left[(1+\delta k)^{-1} - P(T, T+\delta) \right]^+}{P(T, T+\delta)} | \mathcal{F}_t \right\}, \qquad (3)
\end{aligned}$$

where the last line follows from the law of iterative expectations. Changing to the forward measure $\mathbb{Q}_T$ associated to the numeraire $P(t, T)$, then equation (3) yields

$$\begin{aligned}
& \delta c_t [E(T, T+\delta); k; T+\delta] \\
= \;& (1+\delta k) P(t, T) \mathbb{E}_{\mathbb{Q}_T} \left\{ \frac{\left[(1+\delta k)^{-1} - P(T, T+\delta) \right]^+}{P(T, T)} | \mathcal{F}_t \right\} \\
= \;& (1+\delta k) p_t \left[P(T, T+\delta); (1+\delta k)^{-1}; T \right], \qquad (4)
\end{aligned}$$

i.e. a European-style put with contract size $(1+\delta k)$, strike $(1+\delta k)^{-1}$, expiry date at time T, and on a zero-coupon bond with maturity at time $T+\delta$. Such put option is given, under the Cox, Ingersoll and Ross (1985) model, by equation (381) of the handouts.

(a) Via equação (64) dos handouts, o valor da put Europeia (com $\beta > 2$) é dado por:

$$c_t(S,X,T) = S_t e^{-q(T-t)} Q_{\chi^2\left(\frac{2}{\beta-2}, 2\kappa X^{2-\beta}\right)}(2x) - X e^{-r(T-t)} F_{\chi^2\left(2+\frac{2}{\beta-2}, 2x\right)}\left(2\kappa X^{2-\beta}\right), \tag{5}$$

$$\kappa := \frac{2(r-q)}{(2-\beta)\,\delta^2\left[e^{(2-\beta)(r-q)(T-t)}-1\right]}, \tag{6}$$

e

$$x := \kappa S_t^{2-\beta} e^{(2-\beta)(r-q)(T-t)}. \tag{7}$$

Sendo a volatilidade da taxa de rentabilidade do activo subjacente é igual a 30% ao ano, então, e via equação (2) dos handouts,

$$\delta = \frac{30\%}{(100)^{\frac{4-2}{2}}} = 0.003.$$

Retomando as equações (6) e (7),

$$\kappa = \frac{2(1\%-3\%)}{(2-4)(0.003)^2\left[e^{(2-4)(1\%-3\%)\times 0.5}-1\right]} \cong 110003.7037,$$

e

$$x = 110003.7037 \times (100)^{2-4}\, e^{(2-4)(1\%-3\%)\times 0.5} \cong 11.22259259.$$

Por outro lado, sendo a opção ATM-*forward*, tal significa que o *strike* é igual ao preço *forward* do activo subjacente para a data de vencimento da opção:

$$\begin{aligned} X &= S_t e^{-(r-q)(T-t)} \\ &= 100 \times e^{(1\%-3\%)\times 0.5} \cong 99.0050. \end{aligned}$$

Substituindo na equação (5), então

$$\begin{aligned} c_t &= 100 \times e^{-3\%\times 0.5} \times Q_{\chi^2\left(\frac{2}{4-2}, 2\times 110003.7037\times 99.0050^{2-4}\right)}(2\times 11.22259259) \\ &\quad -99.0050 \times e^{-1\%\times 0.5} \times F_{\chi^2\left(2+\frac{2}{4-2}, 2\times 11.22259259\right)}\left(2\times 110003.7037\times 99.0050^{2-4}\right) \\ &= 100 \times e^{-3\%\times 0.5} \times Q_{\chi^2(1,22.44519)}(22.44519) \\ &\quad -99.0050 \times e^{-1\%\times 0.5} \times F_{\chi^2(3,22.44519)}(22.44519). \end{aligned} \tag{8}$$

Via tabela do enunciado, sabemos que

$$F_{\chi^2(3,22.44519)}(22.44519) = 0.41596. \tag{9}$$

A probabilidade $Q_{\chi^2(1,22.44519)}(22.44519)$ pode ser calculada via aproximação de Sankaran, i.e.

$$\begin{aligned} Q_{\chi^2(a,b)}(z) &= 1 - \mathbb{Q}\left(\chi^2(a,b) < z\right) \\ &= 1 - \mathbb{Q}\left\{\left[\frac{\chi^2(a,b)}{a+b}\right]^h < \left(\frac{z}{a+b}\right)^h\right\} \\ &\approx \Phi\left[-\frac{\left(\frac{z}{a+b}\right)^h - \mu_h}{\sigma_h}\right], \end{aligned} \tag{10}$$

onde

$$\mu_h := 1 + h(h-1)\frac{a+2b}{(a+b)^2} - h(h-1)(2-h)(1-3h)\frac{(a+2b)^2}{2(a+b)^4}, \tag{11}$$

$$\sigma_h^2 := h^2\frac{2(a+2b)}{(a+b)^2}\left[1-(1-h)(1-3h)\frac{a+2b}{(a+b)^2}\right], \tag{12}$$

e

$$h := 1 - \frac{2}{3}(a+b)(a+3b)(a+2b)^{-2}. \tag{13}$$

Visto que $a = 1$ e $b = 22.44519$, então

$$\begin{aligned} h &= 1 - \frac{2}{3}(1+22.44519)(1+3\times 22.44519) \\ &\quad (1+2\times 22.44519)^{-2} \\ &\cong 0.492815454, \end{aligned}$$

$$\begin{aligned} \mu_h &= 1 + 0.492815454 \times (0.492815454 - 1)\frac{1+2\times 22.44519}{(1+22.44519)^2} \\ &\quad -0.492815454 \times (0.492815454 - 1)(2 - 0.492815454) \\ &\quad (1 - 3\times 0.492815454)\frac{(1+2\times 22.44519)^2}{2(1+22.44519)^4} \\ &\cong 0.978504653, \end{aligned}$$

e

$$\begin{aligned} \sigma_h^2 &= 0.492815454^2 \times \frac{2(1+2\times 22.44519)}{(1+22.44519)^2} \\ &\quad \left[1-(1-0.492815454)(1-3\times 0.492815454)\frac{1+2\times 22.44519}{(1+22.44519)^2}\right] \\ &\cong 0.04137359. \end{aligned}$$

Utilizando a equação (10),

$$\begin{aligned} Q_{\chi^2(1,22.44519)}(22.44519) &= \Phi\left[-\frac{\left(\frac{22.44519}{1+22.44519}\right)^{0.492815454} - 0.978504653}{\sqrt{0.04137359}}\right] \\ &\cong 0.499523241. \end{aligned} \tag{14}$$

Finalmente, combinando as equações (8), (9) e (14),

$$\begin{aligned} c_t &= 100 \times e^{-3\% \times 0.5} \times 0.499523241 - 99.0050 \times e^{-1\% \times 0.5} \times 0.41596 \\ &\cong EUR8.23199. \end{aligned}$$

(b) Visto tratarem-se de opções ATM-*forward*, então

$$p_t = c_t = EUR8.23199.$$

(a) O payoff terminal de uma range digital é dado por

$$RD_T\left(S, X_a, X_b, T; M\right) = M \mathbb{1}_{\{X_a < S_T < X_b\}}.$$

Consequentemente,

$$\begin{aligned} RD_t\left(S, X_a, X_b, T; M\right) &= Me^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}\left(\mathbb{1}_{\{X_a < S_T < X_b\}} \middle| \mathcal{F}_t\right) \\ &= Me^{-r(T-t)} \mathbb{Q}\left(X_a < S_T < X_b \middle| \mathcal{F}_t\right) \\ &= Me^{-r(T-t)} \left[\mathbb{Q}\left(S_T > X_a \middle| \mathcal{F}_t\right) - \mathbb{Q}_S\left(S_T > X_b \middle| \mathcal{F}_t\right)\right] \end{aligned} \quad (15)$$

Visto que

$$\mathbb{Q}\left(S_T > X \middle| \mathcal{F}_t\right) = P_2\left(S_t, v_t; T, X\right), \quad (16)$$

então combinando as equações (15) e (16),

$$RD_t\left(S, X_a, X_b, T; M\right) = Me^{-r(T-t)} \left[P_2\left(S_t, v_t; T, X_a\right) - P_2\left(S_t, v_t; T, X_b\right)\right]. \quad (17)$$

No caso em apreço

$$\begin{aligned} P_2\left(S_t, v_t; T, X_a = 90\right) &= \frac{1}{2} + \frac{0.21671767}{\pi} \\ &\cong 5.6898E-01 \end{aligned}$$

e

$$\begin{aligned} P_2\left(S_t, v_t; T, X_b = 110\right) &\approx \frac{1}{2} + \frac{-0.64012367}{\pi} \\ &\cong 2.9624E-01. \end{aligned}$$

Portanto,

$$\begin{aligned} RD_0 &= 1 \times e^{-1\% \times 2} \times (0.56898 - 0.29624) \\ &\cong EUR0.267. \end{aligned}$$

(b) A cotação do futuro é dada por:

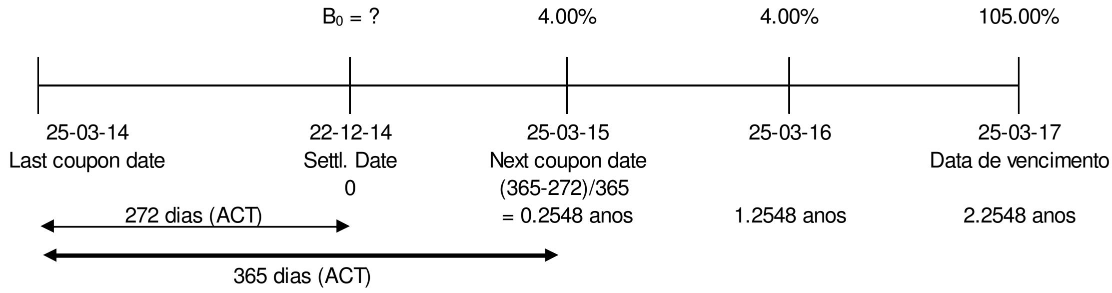
$$\begin{aligned} F_t &= \mathbb{E}_{\mathbb{Q}}\left[S_T \middle| \mathcal{F}_t\right] \\ &= S_t e^{(r-q)(T-t)}, \end{aligned}$$

i.e. corresponde ao preço *forward* visto a taxa de juro ser deperministica. Deste modo,

$$F_t = 100 \times e^{(1\% - 3\%) \times 2} \cong EUR96.08.$$

(a) Settlement date = trade date + 3 dias úteis = 17/12/14 (4^{a} feira) + 5 dias de calendário = 22/12/14.

Pretende-se avaliar uma obrigação com os seguintes cash flows vincendos:



Portanto,

$$\begin{aligned} B_0 &= 4\%P(0, 0.2548) + 4\%P(0, 1.2548) + 105\%P(0, 2.2548) \\ &= 4\%P(0, 0.2548) + 4\% \times 0.9727 + 105\% \times 0.9457. \end{aligned} \tag{18}$$

Relativamente ao factor de desconto a 0.2548 anos, via equações (269) e (270) dos apontamentos,

$$\begin{aligned} B(0.25482) &= \frac{1 - e^{-2\times 0.2548}}{2} \\ &\cong 0.1996, \end{aligned}$$

e

$$\begin{aligned} A(0.25482) &= (0.1996 - 0.25482)\left(3\% - \frac{0.1^2}{2(2)^2}\right) - \frac{0.1^2}{4\times 2} \times (0.1996)^2 \\ &\cong -0.0016. \end{aligned}$$

Portanto,

$$\begin{aligned} P(0, 0.2548) &= \exp(-0.0016 - 0.1996 \times 1\%) \\ &\cong 0.9964. \end{aligned}$$

Retomando a equação (18), então

$$\begin{aligned} B_0 &= 4\% \times 0.9964 + 4\% \times 0.9727 + 105\% \times 0.9457 \\ &\cong 107.172\%. \end{aligned}$$

(b) Utilizando a Proposição 59 dos apontamentos,

$$\begin{aligned} & p_0\left[P(0, 1.2548); 97.527\%; 0.2548\right] \\ = \ & -P(0, 1.2548)\,\Phi\left(-d_1^V\right) + 97.527\%P(0, 0.2548)\,\Phi\left(-d_0^V\right) \\ = \ & -0.9727 \times \Phi\left(-d_1^V\right) + 0.97527 \times 0.9964 \times \Phi\left(-d_0^V\right), \end{aligned}$$

onde

$$
\begin{aligned}
v(0, 0.2548, 1.2548) &= \sqrt{\frac{0.1^2}{2^2}\left[1-e^{-2\times1}\right]^2 \frac{1-e^{-2\times2\times0.2548}}{2\times2}} \\
&\cong 1.728\%,
\end{aligned}
$$

$$
\begin{aligned}
d_1^V &= \frac{\ln\left(\frac{0.9727}{0.97527\times0.9964}\right)+\frac{(1.728\%)^2}{2}}{1.728\%} \\
&\cong 0.063707621,
\end{aligned}
$$

e

$$
\begin{aligned}
d_0^V &= 0.063707621 - 1.728\% \\
&\cong 0.046426375.
\end{aligned}
$$

Portanto,

$$
\begin{aligned}
& p_0\left[P(0, 1.2548); 97.527\%; 0.2548\right] \\
= \; & -0.9727\times\Phi(-0.063707621) + 0.93413\times0.9964\times\Phi(-0.046426375) \\
= \; & -0.9727\times0.474601518 + 0.93413\times0.9964\times0.481485208 \\
\cong \; & 0.625\%. \qquad (19)
\end{aligned}
$$

(c) De acordo com a Proposição 61 dos apontamentos, o valor actual de uma *put* sobre a CBB pode ser decomposto numa carteira de 2 *puts* Europeias sobre PBD:

$$
\begin{aligned}
& p_0(B_t; X = 102.50\%; T = 0.2548) \qquad (20) \\
= \; & 4\%\times p_0\left[P(0, 1.2548); X_1; T = 0.2548\right] \\
& +105\%\times p_0\left[P(0, 2.2548); X_2; T = 0.2548\right].
\end{aligned}
$$

Os *strikes* podem ser obtidos via equação (327) dos apontamentos:

$$
\begin{aligned}
X_1 &= \exp\left[A(1.2548-0.2548) - B(1)\times1.964\%\right] \\
&= \exp(-0.0166 - 0.4323\times1.964\%) \\
&\cong 97.527\%,
\end{aligned}
$$

e

$$
\begin{aligned}
X_2 &= \exp\left[A(2.2548-0.2548) - B(2)\times1.964\%\right] \\
&= \exp(-0.0437 - 0.4908\times1.964\%) \\
&\cong 94.807\%.
\end{aligned}
$$

Portanto,

$$
\begin{aligned}
& p_0(B_t; X = 102.50\%; T = 0.2548) \qquad (21) \\
= \; & 4\%\times p_0\left[P(0, 1.2548); X_1 = 97.527\%; T = 0.2548\right] \\
& +105\%\times p_0\left[P(0, 2.2548); X_2 = 94.807\%; T = 0.2548\right].
\end{aligned}
$$

A primeira *put* já foi avaliada na alínea anterior –vide equação (19):

$$p_0\left[P\left(0,1.2548\right);97.527\%;0.2548\right]\cong 0.625\%. \tag{22}$$

Relativamente à segunda *put*, o enunciado fornece o seguinte valor actual:

$$p_0\left[P\left(0,2.2548\right);X_2=94.807\%;T=0.2548\right]=0.689\%. \tag{23}$$

Combinando as equações (21), (22) e (23),

$$\begin{aligned}&p_0\left(B_t;X=102.50\%;T=0.2548\right)\\=\;&4\%\times 0.625\%+105\%\times 0.689\%\\\cong\;&0.741\%.\end{aligned}$$

(a) Pretende-se avaliar a seguinte opção:

$$c_0\left[P\left(0,1\right);K=\frac{P\left(0,1\right)}{P\left(0,0.5\right)};0.5\right].$$

Via Proposição 68,

$$\begin{aligned}c_0\left[P\left(0,1\right);K=\frac{0.9845}{0.9932};0.5\right]&=P\left(0,1\right)\times F_{\chi^2_{\left(\frac{4\times 2\times 2\%}{0.1^2},\zeta_2\right)}}\left(\frac{r^*}{L_2}\right)\\&-\frac{0.9845}{0.9932}\times P\left(0,0.5\right)\times F_{\chi^2_{(16,\zeta_1)}}\left(\frac{r^*}{L_1}\right),\end{aligned}\tag{24}$$

sendo

$$\begin{aligned}\gamma&=\sqrt{2^2+2\times\left(10\%\right)^2}\\&\cong 2.004993766,\end{aligned}$$

$$\begin{aligned}\zeta_2&=\frac{8r_t\gamma^2e^{\gamma(T_1-t)}}{\sigma^2\left[e^{\gamma(T_1-t)}-1\right]\left\{\gamma\left[e^{\gamma(T_1-t)}+1\right]+\left[k-\sigma^2B\left(T_2-T_1\right)\right]\left[e^{\gamma(T_1-t)}-1\right]\right\}}\\&=\left[8\times 1\%\times\left(2.004993766\right)^2\times e^{2.004993766\times 0.5}\right]\left\{0.1^2\times\left(e^{2.004993766\times 0.5}-1\right)\right.\\&\quad\left[2.004993766\times\left(e^{2.004993766\times 0.5}+1\right)\right.\\&\quad\left.\left.+\left(2-0.1^2\times(-0.3160)\right)\left(e^{2.004993766\times 0.5}-1\right)\right]\right\}^{-1}\\&\cong 4.65039831,\end{aligned}$$

$$\begin{aligned}L_2&=\frac{\sigma^2}{2}\frac{e^{\gamma(T_1-t)}-1}{\gamma\left[e^{\gamma(T_1-t)}+1\right]+\left[k-\sigma^2B\left(T_2-T_1\right)\right]\left[e^{\gamma(T_1-t)}-1\right]}\\&=\frac{\frac{0.1^2}{2}\times\left(e^{2.004993766\times 0.5}-1\right)}{2.004993766\left(e^{2.004993766\times 0.5}+1\right)+\left(2-0.1^2\left(-0.3160\right)\right)\left(e^{2.004993766\times 0.5}-1\right)}\\&\cong 0.000789555,\end{aligned}$$

e

$$\begin{aligned} r^* &= \frac{\ln(K) - A(T_2 - T_1)}{B(T_2 - T_1)} \\ &= \frac{\ln(0.9912) - A(1 - 0.5)}{B(0.5)} \\ &= \frac{\ln(0.9912) - (-0.0037)}{-0.3160} \\ &\cong 1.631\%. \end{aligned}$$

Portanto,

$$\begin{aligned} & c_0 \left[P(0,1); K = 99.12\%; 0.5 \right] \\ = \; & 0.9845 \times F_{\chi^2_{(16, 4.65039831)}} \left(\frac{1.631\%}{0.000789555} \cong 20.65671293 \right) \\ & -\frac{0.9845}{0.9932} \times 0.9932 \times F_{\chi^2_{(16, 4.652719864)}} \left(\frac{1.631\%}{0.000789949} \cong 20.64640591 \right). \end{aligned} \tag{25}$$

Com base no enunciado, podemos calcular as duas probabilidades contidas na equação anterior:

$$F_{\chi^2_{(16, 4.65039831)}} (20.65671293) = 0.544609, \tag{26}$$

e

$$F_{\chi^2_{(16, 4.652719864)}} (20.64640591) = 0.543905. \tag{27}$$

Finalmente, combinando as equações (25), (26) e (27),

$$\begin{aligned} c_0 \left[P(0,1); K = 99.12\%; 0.5 \right] &= 0.9845 \times 0.544609 \\ &\quad -0.9845 \times 0.543905 \\ &\cong 0.00069381. \end{aligned}$$

(b) Seja $t = 0$, $T_1 = 1$ anos, $T_2 = 1.5$ anos, $K_e = 1.37\%$, $M = EUR1M$ e $E(T_1, T_2)$ o valor daqui a T_1 anos da Euribor em vigor entre os momentos T_1 e T_2. Consequentemente, o payoff terminal da opção, daqui a 0.5 anos, é dado por

$$V_{T_1} = M \times \mathbb{1}_{\{E(T_1, T_2) > K_e\}}. \tag{28}$$

Visto que

$$P(T_1, T_2) = \frac{1}{1 + E(T_1, T_2) \times (T_2 - T_1)},$$

então

$$E(T_1, T_2) = \left[\frac{1}{P(T_1, T_2)} - 1 \right] \times \frac{1}{T_2 - T_1}. \tag{29}$$

Combinando as equações (28) e (29), então

$$\begin{aligned} V_{T_1} &= M \times \mathbb{1}_{\left\{\frac{1}{P(T_1,T_2)}-1>K_e(T_2-T_1)\right\}} \\ &= M \times \mathbb{1}_{\left\{P(T_1,T_2)<[1+K_e(T_2-T_1)]^{-1}\right\}}. \end{aligned} \tag{30}$$

Visto que

$$P(T_1, T_2) = \exp\left[A(T_2 - T_1) + B(T_2 - T_1) r_{T_1}\right],$$

então a equação (30) pode ser reescrita como

$$\begin{aligned} V_{T_1} &= M \times \mathbb{1}_{\left\{A(T_2-T_1)+B(T_2-T_1)r_{T_1}<\ln[1+K_e(T_2-T_1)]^{-1}\right\}} \\ &= M \times \mathbb{1}_{\left\{B(T_2-T_1)r_{T_1}<-A(T_2-T_1)-\ln[1+K_e(T_2-T_1)]\right\}} \\ &= M \times \mathbb{1}_{\left\{r_{T_1}>\frac{-A(T_2-T_1)-\ln[1+K_e(T_2-T_1)]}{B(T_2-T_1)}\right\}}, \end{aligned} \tag{31}$$

uma vez que $B(T_2 - T_1) < 0$.
Consequentemente, o valor actual da opção é dado por

$$\begin{aligned} V_t &= P(t, T_1) \times \mathbb{E}_{\mathbb{Q}_{T_1}}\left[\left.\frac{M \times \mathbb{1}_{\left\{r_{T_1}>-\frac{A(T_2-T_1)+\ln[1+K_e(T_2-T_1)]}{B(T_2-T_1)}\right\}}}{P(T_1, T_1)}\right| \mathcal{F}_t\right] \\ &= M \times P(t, T_1) \times \mathbb{Q}_{T_1}\left[r_{T_1} > -\left.\frac{A(T_2 - T_1) + \ln\left[1 + K_e(T_2 - T_1)\right]}{B(T_2 - T_1)}\right| \mathcal{F}_t\right] \end{aligned} \tag{32}$$

Visto que

$$\frac{r_{T_1}}{L_1} \overset{\mathbb{Q}_{T_1}}{\sim} \chi^2\left(\frac{4k\theta}{\sigma^2}, \zeta_1\right),$$

então

$$\begin{aligned} V_t &= M \times P(t, T_1) \times \mathbb{Q}_{T_1}\left[\frac{r_{T_1}}{L_1} > -\left.\frac{A(T_2 - T_1) + \ln\left[1 + K_e(T_2 - T_1)\right]}{B(T_2 - T_1) L_1}\right| \mathcal{F}_t\right] \\ &= M \times P(t, T_1) \times Q_{\chi^2\left(\frac{4k\theta}{\sigma^2}, \zeta_1\right)}\left[-\frac{A(T_2 - T_1) + \ln\left[1 + K_e(T_2 - T_1)\right]}{B(T_2 - T_1) L_1}\right]. \end{aligned} \tag{33}$$

Para os dados do problema em apreço,

$$\begin{aligned} V_t &= EUR1M \times P(0, 1) \times Q_{\chi^2(16,4.652719864)}\left[-\frac{A(1.5 - 1) + \ln\left[1 + K_e(T_2 - T_1)\right]}{B(1.5 - 1) \times 0.000789949}\right] \\ &= EUR1M \times 0.9845 \times Q_{\chi^2(16,4.652719864)}\left[-\frac{-0.0037 + \ln\left[1 + 1.37\% \times 0.5\right]}{-0.3160 \times 0.000789949}\right] \\ &= EUR1M \times 0.9845 \times Q_{\chi^2(16,4.652719864)}(12.61314043) \\ &= EUR1M \times 0.9845 \times (1 - 0.115051) \\ &\cong EUR871,190.12. \end{aligned}$$

Referências

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