Modelos de Estrutura Temporal de Taxas de Juro
Mestrado em Matemática Financeira 17/18
IBS e FCUL
Exame 1ª Época - Resolução

18/Dec/18 Duration: 3h

1. (a) Denoting by V_t the (time t) *fair value* of the asset-or-nothing put, and by $\mathbb{Q}_j$ the forward martingale measure associated to the numeraire $P(t,T_j)$, for $j=1,2$, then

$$\begin{aligned} V_t &= P(t,T_1)\,\mathbb{E}_{\mathbb{Q}_1}\left[\left.\frac{P(T_1,T_2)\,\mathbb{1}_{\{P(T_1,T_2)<K\}}}{P(T_1,T_1)}\right|\mathcal{F}_t\right] \\ &= P(t,T_2)\,\mathbb{E}_{\mathbb{Q}_2}\left[\left.\frac{P(T_1,T_2)\,\mathbb{1}_{\{P(T_1,T_2)<K\}}}{P(T_1,T_2)}\right|\mathcal{F}_t\right] \\ &= P(t,T_2)\,\mathbb{Q}_2\left[P(T_1,T_2)<K|\,\mathcal{F}_t\right] \\ &= P(t,T_2)\,\mathbb{Q}_2\left[P(T_1,T_1,T_2)<K|\,\mathcal{F}_t\right], \end{aligned} \tag{1}$$

where the second equality follows from the change of numeraire technique.

Using equation (321) of the handouts, then

$$P(T_1,T_1,T_2)=\frac{P(t,T_2)}{P(t,T_1)}\exp\left[\frac{1}{2}v^2(t,T_1,T_2)-Y^{\mathbb{Q}_2}\right], \tag{2}$$

where

$$Y^{\mathbb{Q}_2}:=\rho\int_t^{T_1}\left[B(T_2-s)-B(T_1-s)\right]dW_s^{\mathbb{Q}_2}\bigg|\,\mathcal{F}_t\overset{\mathbb{Q}_2}{\sim}N^1\left(0,v^2(t,T_1,T_2)\right).$$

Combining equations (1) and (2), then

$$\begin{aligned} V_t &= P(t,T_2)\,\mathbb{Q}_2\left\{\frac{1}{2}v^2(t,T_1,T_2)-Y^{\mathbb{Q}_2}<\ln\left[\frac{KP(t,T_1)}{P(t,T_2)}\right]|\mathcal{F}_t\right\} \\ &= P(t,T_2)\,\mathbb{Q}_2\left\{Y^{\mathbb{Q}_2}>-\ln\left[\frac{KP(t,T_1)}{P(t,T_2)}\right]+\frac{1}{2}v^2(t,T_1,T_2)\,|\mathcal{F}_t\right\} \\ &= P(t,T_2)\,\Phi\left\{-\frac{\ln\left[\frac{P(t,T_2)}{KP(t,T_1)}\right]+\frac{1}{2}v^2(t,T_1,T_2)}{v(t,T_1,T_2)}\right\}. \end{aligned} \tag{3}$$

(b) Using risk neutral valuation but discounting back to the determination time (and not to the valuation date), then

$$
\begin{aligned}
p^{rf}_{T_1}\left(S,\beta S_{T_1},T_2\right) &= e^{-r(T_2-T_1)}\mathbb{E}_{\mathbb{Q}}\left[\left.p^{rf}_{T_2}\left(S,\beta S_{T_1},T_2\right)\right|\mathcal{F}_{T_1}\right] \\
&= e^{-r(T_2-T_1)}\mathbb{E}_{\mathbb{Q}}\left[\left.\left(\beta-\frac{S_{T_2}}{S_{T_1}}\right)^+\right|\mathcal{F}_{T_1}\right] \\
&= \frac{1}{S_{T_1}}e^{-r(T_2-T_1)}\mathbb{E}_{\mathbb{Q}}\left[\left.\left(\beta S_{T_1}-S_{T_2}\right)^+\right|\mathcal{F}_{T_1}\right] \\
&= \frac{1}{S_{T_1}}p_{T_1}\left(S,\beta S_{T_1},T_2\right). \qquad (4)
\end{aligned}
$$

Using equation (153) of the handouts, and since $P_j\left(S_t,v_t;T,X\right)$ are homogeneous functions of degree zero in the spot and the strike (under the Heston (1993) model), then

$$
\begin{aligned}
p_{T_1}\left(S,\beta S_{T_1},T_2\right) &= -S_{T_1}e^{-q(T_2-T_1)}\left[1-P_1\left(S_{T_1},v_{T_1};T_2,\beta S_{T_1}\right)\right] \\
&\quad +e^{-r(T_2-T_1)}\beta S_{T_1}\left[1-P_2\left(S_{T_1},v_{T_1};T_2,\beta S_{T_1}\right)\right] \\
&= S_{T_1}\left\{-e^{-q(T_2-T_1)}\left[1-P_1\left(1,v_{T_1};T_2,\beta\right)\right]\right. \\
&\quad \left.+e^{-r(T_2-T_1)}\beta\left[1-P_2\left(1,v_{T_1};T_2,\beta\right)\right]\right\} \\
&= S_{T_1}p_{T_1}\left(1,\beta,T_2\right). \qquad (5)
\end{aligned}
$$

Combining equations (4) and (5), and since $p_{T_1}\left(1,\beta,T_2\right)$ is a constant (as we are assuming that the interest rate, the dividend yield and the volatility are all constant), then

$$
\begin{aligned}
p^{rf}_{t}\left(S,\beta S_{T_1},T_2\right) &= e^{-r(T_1-t)}\mathbb{E}_{\mathbb{Q}}\left[\left.p^{rf}_{T_1}\left(S,\beta S_{T_1},T_2\right)\right|\mathcal{F}_{t}\right] \\
&= e^{-r(T_1-t)}\mathbb{E}_{\mathbb{Q}}\left[\left.\frac{1}{S_{T_1}}p_{T_1}\left(S,\beta S_{T_1},T_2\right)\right|\mathcal{F}_{t}\right] \\
&= e^{-r(T_1-t)}\mathbb{E}_{\mathbb{Q}}\left[\left.\frac{1}{S_{T_1}}S_{T_1}p_{T_1}\left(1,\beta,T_2\right)\right|\mathcal{F}_{t}\right] \\
&= e^{-r(T_1-t)}p_{T_1}\left(1,\beta,T_2\right). \qquad (6)
\end{aligned}
$$

(c) Since $r(t) = \sum_{j=1}^{n} Y_j(t)$, then

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[\exp\left(\mu\int_t^T r(u)\,du\right)\middle|\mathcal{F}_t\right] &= \mathbb{E}_{\mathbb{Q}}\left[\exp\left(\mu\int_t^T \sum_{j=1}^{n} Y_j(u)\,du\right)\middle|\mathcal{F}_t\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\exp\left(\mu\sum_{j=1}^{n}\int_t^T Y_j(u)\,du\right)\middle|\mathcal{F}_t\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\prod_{j=1}^{n}\exp\left(\mu\int_t^T Y_j(u)\,du\right)\middle|\mathcal{F}_t\right] \\
&= \prod_{j=1}^{n}\mathbb{E}_{\mathbb{Q}}\left[\exp\left(\mu\int_t^T Y_j(u)\,du\right)\middle|\mathcal{F}_t\right], \qquad (7)
\end{aligned}$$

where the last equality arises because the processes $\{Y_j(T)\}_{j=1,\ldots,n}$ are independent.

Using Proposition 66 of the handouts, then

$$\mathbb{E}_{\mathbb{Q}}\left[\exp\left(\mu\int_t^T r(u)\,du\right)\middle|\mathcal{F}_t\right] = \prod_{j=1}^{n}\exp\left[\phi_{0,-\mu}(T-t) - r(t)\,\psi_{0,-\mu}(T-t)\right], \quad (8)$$

where

$$\phi_{0,-\mu}(T-t) := \frac{2k_j\theta_j}{\sigma_j^2}\ln\left[\frac{2h_j e^{\frac{\left(k_j+h_j\right)(T-t)}{2}}}{h_j - k_j + (k_j+h_j)\,e^{h_j(T-t)}}\right],$$

and

$$\psi_{0,-\mu}(T-t) := \frac{-2\mu\left[e^{h_j(T-t)}-1\right]}{h_j - k_j + (k_j+h_j)\,e^{h_j(T-t)}},$$

with $h_j := \sqrt{k_j^2 + 2\sigma_j^2\mu}$.

(a) Using equation (58) of the handouts, the fair value of the European-style call (with $\beta < 2$) is given by:

$$c_t(S_t, X, T) = S_t e^{-q(T-t)} Q_{\chi^2\left(2+\frac{2}{2-\beta},2x\right)}\left(2\kappa X^{2-\beta}\right) - X e^{-r(T-t)}\left[1 - Q_{\chi^2\left(\frac{2}{2-\beta},2\kappa X^{2-\beta}\right)}(2x)\right], \quad (9)$$

where

$$\kappa := \frac{2(r-q)}{(2-\beta)\,\delta^2\left[e^{(2-\beta)(r-q)(T-t)}-1\right]}, \quad (10)$$

and

$$x := \kappa S_t^{2-\beta} e^{(2-\beta)(r-q)(T-t)}. \quad (11)$$

Since the (annualized) standard deviation of stock returns is equal to 30% per year, then, and using equation (2) of the handouts,

$$\delta = \frac{30\%}{(100)^{\frac{-2-2}{2}}} = 3,000.$$

Using equations (10) and (11),

$$\kappa = \frac{2\,(1\% - 3\%)}{(2+2)\,(3,000)^2\left[e^{(2+2)(1\%-3\%)\times 0.5} - 1\right]} \cong 2.8337E - 08,$$

and

$$x = (2.8337E - 08) \times (100)^{2+2}\, e^{(2+2)(1\%-3\%)\times 0.5} \cong 2.722592583.$$

Hence, equation (9) yields

$$\begin{aligned} c_t &= 100 \times e^{-3\%\times 0.5} \times Q_{\chi^2\left(2+\frac{2}{2+2}, 2\times 2.722592583\right)}\left(2 \times (2.8337E-08) \times 95^{2+2}\right) \\ &\quad -95 \times e^{-1\%\times 0.5} \times F_{\chi^2\left(\frac{2}{2+2}, 2\times(2.8337E-08)\times 95^{2+2}\right)}\left(2 \times 2.722592583\right) \\ &= 100 \times e^{-3\%\times 0.5} \times Q_{\chi^2(2.5, 5.445185165)}\left(4.616138739\right) \\ &\quad -95 \times e^{-1\%\times 0.5} \times F_{\chi^2(0.5, 4.616138739)}\left(5.445185165\right). \end{aligned} \tag{12}$$

From the table provided in the exam, we know that

$$F_{\chi^2(0.5, 4.616138739)}\left(5.445185165\right) = 0.62220. \tag{13}$$

The probability $Q_{\chi^2(2.5, 5.445185165)}(4.616138739)$ can be computed using the Sankaran approximation, i.e.

$$\begin{aligned} Q_{\chi^2(a,b)}(z) &= \mathbb{Q}\left(\chi^2(a,b) > z\right) \\ &= \mathbb{Q}\left\{\left[\frac{\chi^2(a,b)}{a+b}\right]^h > \left(\frac{z}{a+b}\right)^h\right\} \\ &\approx \Phi\left[-\frac{\left(\frac{z}{a+b}\right)^h - \mu_h}{\sigma_h}\right], \end{aligned} \tag{14}$$

where

$$\mu_h := 1 + h(h-1)\frac{a+2b}{(a+b)^2} - h(h-1)(2-h)(1-3h)\frac{(a+2b)^2}{2(a+b)^4}, \tag{15}$$

$$\sigma_h^2 := h^2 \frac{2(a+2b)}{(a+b)^2}\left[1 - (1-h)(1-3h)\frac{a+2b}{(a+b)^2}\right], \tag{16}$$

and

$$h := 1 - \frac{2}{3}(a+b)(a+3b)(a+2b)^{-2}. \tag{17}$$

Since $a = 2.5$ and $b = 5.445185165$, then

$$\begin{aligned} h &= 1 - \frac{2}{3}(2.5 + 5.445185165)(2.5 + 3 \times 5.445185165) \\ &\quad (2.5 + 2 \times 5.445185165)^{-2} \\ &\cong 0.443575787, \end{aligned}$$

$$\begin{aligned} \mu_h &= 1 + 0.443575787 \times (0.443575787 - 1)\frac{2.5 + 2 \times 5.445185165}{(2.5 + 5.445185165)^2} \\ &\quad -0.443575787 \times (0.443575787 - 1)(2 - 0.443575787) \\ &\quad (1 - 3 \times 0.443575787)\frac{(2.5 + 2 \times 5.445185165)^2}{2(2.5 + 5.445185165)^4} \\ &\cong 0.944786651, \end{aligned}$$

and

$$\begin{aligned} \sigma_h^2 &= 0.443575787^2 \times \frac{2(2.5 + 2 \times 5.445185165)}{(2.5 + 5.445185165)^2} \\ &\quad \left[1 - (1 - 0.443575787)(1 - 3 \times 0.443575787)\frac{2.5 + 2 \times 5.445185165}{(2.5 + 5.445185165)^2}\right] \\ &\cong 0.086732252. \end{aligned}$$

Using equation (14), then

$$\begin{aligned} & Q_{\chi^2(2.5, 5.445185165)}(4.616138739) \\ &= \Phi\left[-\frac{\left(\frac{4.616138739}{2.5 + 5.445185165}\right)^{0.443575787} - 0.944786651}{\sqrt{0.086732252}}\right] \\ &\cong 0.705175569. \end{aligned} \tag{18}$$

Finally, combining equations (12), (13) and (18), then

$$\begin{aligned} c_t &= 100 \times e^{-3\% \times 0.5} \times 0.705175569 - 95 \times e^{-1\% \times 0.5} \times 0.62220 \\ &\cong EUR10.65393. \end{aligned}$$

(b) The terminal payoff of an asset-or-nothing call is equal to

$$ANC_T(S, X, T) = S_T \mathbb{1}_{\{S_T > X\}}.$$

Therefore,

$$\begin{aligned} ANC_t(S, X, T) &= e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}\left(S_T \mathbb{1}_{\{S_T > X\}} \middle| \mathcal{F}_t\right) \\ &= S_t e^{qt} \mathbb{E}_{\mathbb{Q}_S}\left(\frac{S_T \mathbb{1}_{\{S_T > X\}}}{S_T e^{qT}} \middle| \mathcal{F}_t\right) \\ &= S_t e^{-q(T-t)} \mathbb{Q}_S(S_T > X | \mathcal{F}_t). \end{aligned} \tag{19}$$

Since

$$\mathbb{Q}_S\left(S_T > X \middle| \mathcal{F}_t\right) = Q_{\chi^2\left(2+\frac{2}{2-\beta},2x\right)}\left(2\kappa X^{2-\beta}\right),$$

then equations (13) and (19) yield:

$$\begin{aligned} ANC_t\left(S,X,T\right) &= 100 \times e^{-3\%\times 0.5} \times Q_{\chi^2(2.5,5.445185165)}\left(4.616138739\right) \\ &= 100 \times e^{-3\%\times 0.5} \times 0.705175569 \\ &\cong EUR69.46768729. \end{aligned}$$

For a contract size of 100 shares, the fair value of the asset-or-nothing call is equal to

$$EUR69.46768729 \times 100 = EUR6,946.768729.$$

(a) Using Proposition 22 of the handouts,

$$\begin{aligned} p_0 &= -100 \times e^{-3\%\times 0.5} \times \left[1 - P_1\left(S_t = 100, v_t = 0.06; T = 0.5, X = 95\right)\right] \\ &\quad + e^{-1\%\times 0.5} \times 95 \times \left[1 - P_2\left(S_t = 100, v_t = 0.06; T = 0.5, X = 95\right)\right]. \end{aligned} \tag{20}$$

Using equations (173) and (174) of the handouts:

$$\begin{aligned} P_1\left(S_t = 100, v_t = 0.06; T = 0.5, X = 95\right) &\approx \frac{1}{2} + \frac{0.44103501}{\pi} \\ &\cong 0.640386, \end{aligned} \tag{21}$$

and

$$\begin{aligned} P_2\left(S_t = 100, v_t = 0.06; T = 0.5, X = 95\right) &\approx \frac{1}{2} + \frac{0.24515768}{\pi} \\ &\cong 0.578036. \end{aligned} \tag{22}$$

Combining equations (20), (21) and (22), then

$$\begin{aligned} p_0 &= -100 \times e^{-3\%\times 0.5} \times (1 - 0.640386) + e^{-1\%\times 0.5} \times 95 \times (1 - 0.578036) \\ &\cong EUR8.4456. \end{aligned}$$

(b) The fair value of the volatility derivative is:

$$V_0 = e^{-1\%\times 0.5} \times \mathbb{E}_{\mathbb{Q}}\left[e^{v_T} \middle| \mathcal{F}_t\right], \tag{23}$$

with $t = 0$ and $T = 0.5$.
The expectation on the RHS of equation (23) is just the Laplace transform of the square-root process

$$dv_t = k\left(\theta - v_t\right)dt + \sigma\sqrt{v_t}dW_2^{\mathbb{Q}}\left(t\right),$$

and, therefore, Proposition 4 of the handouts implies that

$$\mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\lambda v_T\right) \middle| \mathcal{F}_t\right] = \frac{\exp\left(-\frac{\lambda L f}{1+2\lambda L}\right)}{\left(1+2\lambda L\right)^{\frac{2k\theta}{\sigma^2}}}, \tag{24}$$

where

$$L := \frac{\sigma^2 \left[1 - e^{-k(T-t)}\right]}{4k}, \tag{25}$$

$$f := \frac{4 v_t k}{\sigma^2 \left[e^{k(T-t)} - 1\right]}, \tag{26}$$

and $\lambda = -1$.

Hence,

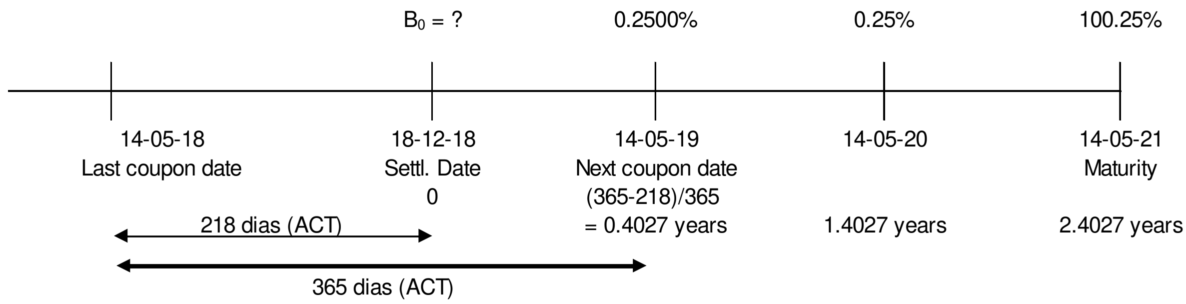
$$L = \frac{0.1^2 \times \left[1 - e^{-2\times 0.5}\right]}{4 \times 2} \cong 0.000790151,$$

$$f = \frac{4 \times 0.06 \times 2}{0.1^2 \times \left[e^{2\times 0.5} - 1\right]} \cong 27.93488193,$$

and

$$V_0 = e^{-1\% \times 0.5} \times \frac{\exp\left(\frac{0.000790151 \times 27.93488193}{1 - 2\times 0.000790151}\right)}{(1 - 2 \times 0.000790151)^{\frac{2\times 2\times 4\%}{0.1^2}}} \cong 1.043324788.$$

(a) The purpose is to price a bond with the following future cash flows:



Therefore,

$$\begin{aligned} B_0 &= 0.25\% \times P(0, 0.4027) + 0.25\% \times P(0, 1.4027) + 100.25\% \times P(0, 2.4027) \\ &= 0.25\% \times P(0, 0.4027) + 0.25\% \times 0.9751 + 100.25\% \times 0.9561. \end{aligned} \tag{27}$$

Concerning the discount factor for the maturity of 0.3068 years, equations (269) and (270) of the handouts imply that

$$\begin{aligned} B(0.4027) &= \frac{1 - e^{-4\times 0.4027}}{4} \\ &\cong 0.2001, \end{aligned}$$

and

$$\begin{aligned} A(0.4027) &= (0.2001 - 0.4027)\left(2\% - \frac{0.1^2}{2(4)^2}\right) - \frac{0.1^2}{4 \times 4} \times (0.2001)^2 \\ &\cong -0.0040. \end{aligned}$$

Hence,

$$\begin{aligned} P(0, 0.4027) &= \exp(-0.0040 - 0.2001 \times 1\%) \\ &\cong 0.9940. \end{aligned}$$

Recalling equation (27), then

$$\begin{aligned} B_0 &= 0.25\% \times 0.9940 + 0.25\% \times 0.9751 + 100.25\% \times 0.9561 \\ &\cong 96.339\%. \end{aligned}$$

(b) On 18/12/18, the accrued interest amount is equal to

$$AI_0 = 0.25\% \times \frac{218}{365} \cong 0.149\%.$$

Hence, the gross prices are:

$$GP_0^{bid} = 96.10\% + 0.149\% = 96.249\% < 96.339\%,$$

i.e. we should not sell the bond, and

$$GP_0^{bid} = 96.15\% + 0.149\% = 96.299\% < 96.339\%,$$

i.e. we should buy the bond.

(c) Using Proposition 59 of the handouts,

$$\begin{aligned} &p_0 [P(0, 2.4027); K = 96.513\%; 0.4027] \\ = \ & -P(0, 2.4027) \times \Phi\left(-d_1^V\right) + 0.96513 \times P(0, 0.4027) \times \Phi\left(-d_0^V\right) \\ = \ & -0.9561 \times \Phi\left(-d_1^V\right) + 0.96513 \times 0.9940 \times \Phi\left(-d_0^V\right), \end{aligned}$$

where

$$\begin{aligned} v(0, 0.4027, 2.4027) &= \sqrt{\frac{0.1^2}{4^2}\left[1 - e^{-4\times 2}\right]^2 \frac{1 - e^{-2\times 4\times 0.4027}}{2\times 4}} \\ &\cong 0.866\%, \end{aligned}$$

$$\begin{aligned} d_1^V &= \frac{\ln\left(\frac{0.9561}{0.96513\times 0.9940}\right) + \frac{(0.866\%)^2}{2}}{0.866\%} \\ &\cong -0.389437691, \end{aligned}$$

and

$$\begin{aligned} d_0^V &= -0.389437691 - 0.866\% \\ &\cong -0.398095587. \end{aligned}$$

Therefore,

$$\begin{aligned} &p_0 [P(0, 2.4027); K = 96.513\%; 0.4027] \\ = \ & -0.9561 \times \Phi(0.389437691) + 0.96513 \times 0.9940 \times \Phi(0.398095587) \\ = \ & -0.9561 \times 0.651523802 + 0.96513 \times 0.9940 \times 0.654720136 \\ \cong \ & 0.519\%. \end{aligned} \tag{28}$$

(d) Using Proposition 61 of the handouts, the fair value of a European-style *put* on a CBB can be decomposed into a portfolio of 2 European-style *puts* on PBDs:

$$\begin{aligned} & p_0\left(B_0; X = 97\%; T = 0.4027\right) \\ = \; & 0.25\% \times p_0\left[P\left(0, 1.4027\right); X_1; T = 0.4027\right] \\ & +100.25\% \times p_0\left[P\left(0, 2.4027\right); X_2; T = 0.4027\right]. \end{aligned} \tag{29}$$

The *strikes* can be obtained through equation (327) of the handouts:

$$\begin{aligned} X_1 &= \exp\left[A\left(1.4027 - 0.4027\right) - B\left(1\right) \times 1\%\right] \\ &= \exp\left(-0.0149 - 0.2454 \times 1\%\right) \\ &\cong 98.425\%, \end{aligned}$$

and

$$\begin{aligned} X_2 &= \exp\left[A\left(2.4027 - 0.4027\right) - B\left(2\right) \times 1\%\right] \\ &= \exp\left(-0.0345 - 0.2499 \times 1\%\right) \\ &\cong 96.513\%. \end{aligned}$$

Hence,

$$\begin{aligned} & p_0\left(B_0; X = 97\%; T = 0.4027\right) \\ = \; & 0.25\% \times p_0\left[P\left(0, 1.4027\right); X_1 = 98.425\%; T = 0.4027\right] \\ & +100.25\% \times p_0\left[P\left(0, 2.4027\right); X_2 = 96.513\%; T = 0.4027\right]. \end{aligned} \tag{30}$$

The second *put* was already priced in the previous question—please see equation (28):

$$p_0\left[P\left(0, 2.4027\right); K = 96.513\%; 0.4027\right] \cong 0.519\%. \tag{31}$$

Concerning the first *put*, the exam provides the following market price:

$$p_0\left[P\left(0, 1.4027\right); X_1 = 98.425\%; T = 0.4027\right] = 0.520\%. \tag{32}$$

Combining equations (30), (31) and (32),

$$\begin{aligned} & p_0\left(B_0; X = 97\%; T = 0.4027\right) \\ = \; & 0.25\% \times 0.520\% + 100.25\% \times 0.519\% \\ \cong \; & 0.522\%. \end{aligned}$$

(a) The purpose is to compute the 6-month forward rate expected today to prevail in 6-month time (and with semi-annual compounding), i.e. $f\left(0, 6M, 12M\right)$:

$$P\left(0, 12M\right) = P\left(0, 6M\right) \times \left[1 + f\left(0, 6M, 12M\right) \times \frac{6}{12}\right]^{-1}. \tag{33}$$

The 1-year discount function can be obtained from Proposition 64 of the handouts:

$$\gamma = \sqrt{k^2 + 2\sigma^2} = \sqrt{4^2 + 2 \times 0.1^2} \cong 4.0025,$$

$$B(1) = -\frac{2\left(e^{4.0025 \times 1} - 1\right)}{2 \times 4.0025 + (4 + 4.0025)\left(e^{4.0025 \times 1} - 1\right)} \cong -0.24535447,$$

and, since $A(1) = -0.01508895$, then

$$P(0, 12M) = \exp(-0.01508895 - 0.24535447 \times 1\%) \cong 0.98261048.$$

Going back to equation (33), and since $P(0, 6M) = 0.99219314$, then

$$0.98261048 = 0.99219314 \times [1 + f(0, 6M, 12M) \times 0.5]^{-1},$$

i.e.

$$f(0, 6M, 12M) = \left(\frac{0.99219314}{0.98261048} - 1\right) \times 2 \cong 1.950\%.$$

(b) The purpose is to price the following option contract:

$$p_0\left[P(0,2); K = \frac{P(0,2)}{P(0,0.5)}; 0.5\right].$$

Using Proposition 68 of the handouts,

$$\begin{aligned} & p_0\left[P(0,2); K = \frac{0.96320264}{0.99219314} \cong 0.9708; 0.5\right] \qquad (34) \\ = \; & -P(0,2) \times Q_{\chi^2_{\left(\frac{4 \times 4 \times 2\%}{0.1^2}, \zeta_2\right)}}\left(\frac{r^*}{L_2}\right) \\ & +0.9708 \times P(0,0.5) \times F_{\chi^2_{(32,\zeta_1)}}\left(\frac{r^*}{L_1}\right), \end{aligned}$$

where $\gamma \cong 4.0025$, $B(2 - 0.5) = 0.97287476$,

$$\begin{aligned} \zeta_2 &= \frac{8 r_t \gamma^2 e^{\gamma(T_1 - t)}}{\sigma^2 \left[e^{\gamma(T_1-t)} - 1\right]\left\{\gamma\left[e^{\gamma(T_1-t)} + 1\right] + \left[k - \sigma^2 B(T_2 - T_1)\right]\left[e^{\gamma(T_1-t)} - 1\right]\right\}} \\ &= \left[8 \times 1\% \times (4.0025)^2 \times e^{4.0025 \times 0.5}\right]\left\{0.1^2 \times \left(e^{4.0025 \times 0.5} - 1\right)\right. \\ &\quad \left[4.0025 \times \left(e^{4.0025 \times 0.5} + 1\right)\right. \\ &\quad \left.\left. + \left(4 - 0.1^2 \times 0.97287476\right) \times \left(e^{4.0025 \times 0.5} - 1\right)\right]\right\}^{-1} \\ &\cong 2.502904304, \end{aligned}$$

$$\begin{aligned} L_2 &= \frac{\sigma^2}{2} \frac{e^{\gamma(T_1-t)} - 1}{\gamma\left[e^{\gamma(T_1-t)} + 1\right] + \left[k - \sigma^2 B(T_2 - T_1)\right]\left[e^{\gamma(T_1-t)} - 1\right]} \\ &= \frac{\frac{0.1^2}{2} \times \left(e^{4.0025 \times 0.5} - 1\right)}{4.0025 \times \left(e^{4.0025 \times 0.51} + 1\right) + \left(4 - 0.1^2 \times 0.97287476\right) \times \left(e^{4.0025 \times 0.5} - 1\right)} \\ &\cong 0.000540329, \end{aligned}$$

and

$$\begin{aligned} r^* &= \frac{\ln(K) - A(T_2 - T_1)}{B(T_2 - T_1)} \\ &= \frac{\ln(0.9708) - A(2 - 0.5)}{B(1.5)} \\ &= \frac{\ln(0.9708) - (-0.02500687)}{-0.24930456} \\ &\cong 1.864\%. \end{aligned}$$

Therefore,

$$\begin{aligned} & p_0 [P(0,2); K \cong 0.9708; 0.5] \\ = \; & -0.96320264 \times Q_{\chi^2_{(32,2.502904304)}} \left(\frac{1.864\%}{0.000540329} \cong 34.50721453 \right) \\ & +0.9708 \times 0.99219314 \times F_{\chi^2_{(32,2.502904304)}} \left(\frac{1.864\%}{0.000540329} \cong 34.49792033 \right). \end{aligned} \tag{35}$$

Using the table provided in the exam, we can compute the two probabilities contained in the previous equation:

$$Q_{\chi^2_{(32,2.502904304)}} (34.50721453) = 1 - 0.533203407 = 0.466796593, \tag{36}$$

and

$$F_{\chi^2_{(32,2.502904304)}} (34.49792033) = 0.532742659. \tag{37}$$

Finally, combining equations (35), (36) and (37), then

$$\begin{aligned} & p_0 [P(0,2); K \cong 0.9708; 0.5] \\ = \; & -0.96320264 \times 0.466796593 + 0.9708 \times 0.99219314 \times 0.532742659 \\ \cong \; & 0.00044379. \end{aligned}$$

References

Heston, S., 1993, A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, *Review of Financial Studies* 6, 327–343.