

**OPÇÕES EXÓTICAS**  
**MSc MATEMÁTICA FINANCEIRA 2010/11**  
**EXAME - Resolução**

**29/07/11**

**Duração: 2.5 horas**

**CASO 1**

a) Enuncie a estratégia de *static hedging* a adoptar para uma posição curta sobre uma *forward-start call* com vencimento no momento  $T_2$ , com início de vigência no momento  $T_1$  ( $< T_2$ ) e com *strike* igual a 110% da cotação *spot* no momento  $T_1$ .

Consideremos a fórmula de avaliação deste contrato:

$$c_t^f(S_t; 1.1S_{T_1}; T_2) = S_t e^{-q(T_1-t)} c_{T_1}(1; 1.1; T_2).$$

Atendendo a que  $e^{-q(T_1-t)} c_{T_1}(1; 1.1; T_2)$  é uma constante, basta comprar um número  $e^{-q(T_1-t)} c_{T_1}(1; 1.1; T_2)$  de unidades do activo subjacente.

No momento  $T_1$ , o valor da carteira de acções será igual a:

$$[S_{T_1} e^{q(T_1-t)}] e^{-q(T_1-t)} c_{T_1}(1; 1.1; T_2)$$

$$= S_{T_1} c_{T_1}(1; 1.1; T_2)$$

$$= c_{T_1}(S_{T_1}; 1.1S_{T_1}; T_2)$$

$$= c_{T_1}^f(S_{T_1}; 1.1S_{T_1}; T_2)$$

b) Deduza a fórmula de avaliação de uma *European-style up-and-out put* com uma barreira (U) superior ao *strike* (X) e sem *rebate*.

Via página 38 dos apontamentos,

$$UO(-1)_t(S; X; U > X; R = 0; T)$$

$$= e^{-r\tau} \int_{-\infty}^{\ln\left(\frac{X}{S_t}\right)} (X - S_t e^y) \left\{ \phi(y; \mu\tau, \sigma\sqrt{\tau}) - \left(\frac{U}{S_t}\right)^{\frac{2\mu}{\sigma^2}} \phi\left[y; 2\ln\left(\frac{U}{S_t}\right) + \mu\tau, \sigma\sqrt{\tau}\right] \right\} dy.$$

Fazendo a mudança de variável de integração  $z = y - 2 \ln\left(\frac{U}{S_t}\right)$ ,

$$UO(-1)_t(S; X; U > X; R = 0; T)$$

$$\begin{aligned} &= e^{-r\tau} \int_{-\infty}^{\ln\left(\frac{X}{S_t}\right)} (X - S_t e^y) \phi(y; \mu\tau, \sigma\sqrt{\tau}) dy \\ &= \left(\frac{U}{S_t}\right)^{\frac{2\mu}{\sigma^2}} e^{-r\tau} \int_{-\infty}^{\ln\left(\frac{X}{U^2/S_t}\right)} \left[ X - S_t \left(\frac{U}{S_t}\right)^2 e^z \right] \phi(z; \mu\tau, \sigma\sqrt{\tau}) dz \\ &= p_t(S; X; T) - \left(\frac{U}{S_t}\right)^{\frac{2\mu}{\sigma^2}} p_t\left(\frac{U^2}{S_t}; X; T\right). \end{aligned}$$

- c) Enuncie a fórmula de avaliação (no momento “t”) de uma *put* com vencimento no momento “ $T_1$ ”, com *strike* “ $X_1$ ”, e sobre uma *cash-or-nothing call* com vencimento no momento “ $T_2(> T_1)$ ”, com *strike* “ $X_1$ ” e *contract size* “ $M$ ”.

Assumindo os pressupostos do modelo de Black-Scholes (1973), valor da put sobre a digital call é dado por:

$$p_t[D(1)_t(S_t, X_2, T_2; M); X_1; T_1] = e^{-r\tau_1} E_Q \left\{ [X_1 - D(1)_{T_1}(S_{T_1}, X_2, T_2; M)]^+ \middle| F_t \right\}$$

Seja  $\bar{S}$  :

$$X_1 = D(1)_{T_1}(\bar{S}, X_2, T_2; M)$$

$$\Leftrightarrow X_1 = M e^{-r(\tau_2 - \tau_1)} \Phi[d_2^M(\bar{S}, X_2)]$$

Então, e visto que  $\Phi[d_2^M(\bar{S}, X_2)]$  é uma função crescente de  $\bar{S}$ ,

$$p_t[D(1)_t(S_t, X_2, T_2; M); X_1; T_1] = e^{-r\tau_1} \int_0^{\bar{S}} [X_1 - D(1)_{T_1}(S, X_2, T_2; M)] p(S_{T_1} \in dS | F_t).$$

Por outro lado, dado que

$$S_{T_1} = S_t \exp \left[ \left( r - q - \frac{\sigma^2}{2} \right) \tau_1 + \sigma \int_t^{T_1} dW_t^Q \right],$$

então:

$$p_t[D(1)_t(S_t, X_2, T_2; M); X_1; T_1]$$

$$= e^{-r\tau_1} \int_0^{\bar{z}} \left[ X_1 - D(1)_{T_1} \left( S_t \exp \left( \left( r - q - \frac{\sigma^2}{2} \right) \tau_1 + \sigma z \right), X_2, T_2; M \right) \right] \frac{1}{\sqrt{2\pi\tau_1}} \exp \left( -\frac{z^2}{2\tau_1} \right) dz$$

$$= e^{-r\tau_1} \int_0^{\bar{z}} \left\{ X_1 - M e^{-r(\tau_2 - \tau_1)} \Phi \left[ d_2^M \left( S_t \exp \left( \left( r - q - \frac{\sigma^2}{2} \right) \tau_1 + \sigma z \right), X_2 \right) \right] \right\} \frac{1}{\sqrt{2\pi\tau_1}} \exp \left( -\frac{z^2}{2\tau_1} \right) dz$$

$$= e^{-r\tau_1} X_1 \int_0^{\bar{z}} \frac{1}{\sqrt{2\pi\tau_1}} \exp \left( -\frac{z^2}{2\tau_1} \right) dz$$

$$- e^{-r\tau_1} \int_0^{\bar{z}} M e^{-r(\tau_2 - \tau_1)} \Phi \left[ d_2^M \left( S_t \exp \left( \left( r - q - \frac{\sigma^2}{2} \right) \tau_1 + \sigma z \right), X_2 \right) \right] \frac{1}{\sqrt{2\pi\tau_1}} \exp \left( -\frac{z^2}{2\tau_1} \right) dz,$$

sendo

$$\bar{z} = \frac{\ln \left( \frac{\bar{S}}{S_t} \right) - \left( r - q - \frac{\sigma^2}{2} \right) \tau_1}{\sigma}.$$

O 1º termo da anterior expressão é fácil de calcular:

$$\begin{aligned}
& e^{-r\tau_1} X_1 \int_0^{\bar{z}} \frac{1}{\sqrt{2\pi\tau_1}} \exp\left(-\frac{z^2}{2\tau_1}\right) dz \\
&= e^{-r\tau_1} X_1 \Phi\left[\frac{\bar{z}}{\sqrt{\tau_1}}\right] \\
&= e^{-r\tau_1} X_1 \Phi\left[\frac{\ln\left(\frac{\bar{S}}{S_t}\right) - \left(r - q - \frac{\sigma^2}{2}\right)\tau_1}{\sigma\sqrt{\tau_1}}\right] \\
&= e^{-r\tau_1} X_1 \Phi\left[-\frac{\ln\left(\frac{S_t}{\bar{S}}\right) + \left(r - q - \frac{\sigma^2}{2}\right)\tau_1}{\sigma\sqrt{\tau_1}}\right] \\
&= e^{-r\tau_1} X_1 \Phi(-\bar{a}_2),
\end{aligned}$$

sendo

$$\bar{a}_{1,2} = \frac{\ln\left(\frac{S_t}{\bar{S}}\right) + \left(r - q \pm \frac{\sigma^2}{2}\right)\tau_1}{\sigma\sqrt{\tau_1}}.$$

O 2º termo

$$-e^{-r\tau_1} \int_0^{\bar{z}} M e^{-r(\tau_2 - \tau_1)} \Phi\left[d_2^M\left(S_t \exp\left(\left(r - q - \frac{\sigma^2}{2}\right)\tau_1 + \sigma z\right), X_2\right)\right] \frac{1}{\sqrt{2\pi\tau_1}} \exp\left(-\frac{z^2}{2\tau_1}\right) dz$$

é equivalente ao 2º termo no lado direito da equação (53) dos handouts, i.e. ao termo associado ao *strike* da underlying options. Portanto,

$$\begin{aligned}
& -e^{-r\tau_1} \int_0^{\bar{z}} M e^{-r(\tau_2 - \tau_1)} \Phi\left[d_2^M\left(S_t \exp\left(\left(r - q - \frac{\sigma^2}{2}\right)\tau_1 + \sigma z\right), X_2\right)\right] \frac{1}{\sqrt{2\pi\tau_1}} \exp\left(-\frac{z^2}{2\tau_1}\right) dz \\
&= -M e^{-r\tau_2} N_2\left(-\bar{a}_2; b_2; -\sqrt{\frac{\tau_1}{\tau_2}}\right),
\end{aligned}$$

onde  $b_2$  é dado pela equação (55) dos handouts.

Em suma,

$$p_t[D(1)_t(S_t, X_2, T_2; M); X_1; T_1] \\ = e^{-r\tau_1} X_1 \Phi(-\bar{a}_2) - M e^{-r\tau_2} N_2\left(-\bar{a}_2; b_2; -\sqrt{\frac{\tau_1}{\tau_2}}\right).$$

## **CASO 2**

a)

$$r: e^{r \times 6/12} = 1 + 1.8\% \times 6/12 \Rightarrow r = \frac{12}{6} \ln(1 + 1.8\% \times 6/12) \cong 1.792\%.$$

$$B_0 = \frac{100\%}{1 + 1.8\% \times 6/12} + RV_0.$$

$$RV_{6M} = \begin{cases} 10\% \Leftarrow S_{6M} < 2,760 \times 95\% = 2,622 \\ 0\% \Leftarrow ELSE \end{cases}$$

$$= 10\% \times D_{6M}(-1)(M = 1; S; X = 2,622; T = 6M).$$

Portanto,

$$RV_0 = 10\% \times D_0(-1)(M = 1; S; X = 2,622; T = 6M) \\ = 10\% \times e^{-1.792\% \times 0.5} \times \Phi[-d_2^M(2,622)]$$

$$\begin{aligned}
\Phi[-d_2^M(2,622)] &= \Phi\left[-\frac{\ln\left(2,760/2,622\right) + \left(1.792\% - 2\% - \frac{(0.25)^2}{2}\right) \times 0.5}{0.25 \times \sqrt{0.5}}\right] \\
&= \Phi(-0.1959) \\
&\cong \Phi(-0.2) \\
&= 1 - \Phi(0.2) \\
&= 1 - 0.5793 \\
&= 0.4207.
\end{aligned}$$

$$RV_0 = 10\% \times e^{-1.792\% \times 0.5} \times 0.4207 = 4.186\%.$$

$$B_0 = 99.11\% + 4.186\% \cong 103.29\% > 100\% \Rightarrow \text{Investir.}$$

b)

Via proposição 20 e equação (57):

$$\begin{aligned}
&c_0[p_0(S_0; X = 2,760; 0.5y); 162.77; 0.25y] \\
&= -2,760 \times e^{-2\% \times 0.5} \times M\left(-a_1^{**}, -b_1; \sqrt{\frac{0.25}{0.5}}\right) + 2,760 \times e^{-1.792\% \times 0.5} \times M\left(-a_2^{**}, -b_2; \sqrt{0.5}\right) \\
&\quad - 162.77 \times e^{-1.792\% \times 0.25} \times \Phi(-a_2^{**}).
\end{aligned}$$

Visto que:

$$S^{**} = 2,710;$$

$$a_1^{**} = \frac{\ln\left(\frac{2,760}{2,710}\right) + \left(1.792\% - 2\% + \frac{(0.25)^2}{2}\right) \times 0.25}{0.25 \times \sqrt{0.25}} \cong 0.20456849;$$

$$a_2^{**} = 0.20456849 - 0.25 \times \sqrt{0.25} \cong 0.079568;$$

$$b_1 = \frac{\ln\left(\frac{2,760}{2,760}\right) + \left(1.792\% - 2\% + \frac{(0.25)^2}{2}\right) \times 0.5}{0.25 \times \sqrt{0.5}} \cong 0.08250376;$$

$$b_2 = 0.08250376 - 0.25 \times \sqrt{0.5} \cong -0.09427;$$

então:

$$\begin{aligned} & c_0[p_0(S_0; X = 2,760; 0.5y); 162.77; 0.25y] \\ &= -2,760 \times e^{-2\% \times 0.5} \times M(-0.20456849, -0.08250376; 0.707107) \\ &+ 2,760 \times e^{-1.792\% \times 0.5} \times M(-0.079568, 0.09427; 0.707107) \\ &- 162.77 \times e^{-1.792\% \times 0.25} \times \Phi(-0.079568). \end{aligned}$$

Utilizando a tabela de probabilidades,

$$\begin{aligned} & c_0[p_0(S_0; X = 2,760; 0.5y); 162.77; 0.25y] \\ &= -2,760 \times e^{-2\% \times 0.5} \times 0.317972 + 2,760 \times e^{-1.792\% \times 0.5} \times 0.37504 - 162.77 \times e^{-1.792\% \times 0.25} \times 0.46829 \\ &\cong 81.125224. \end{aligned}$$

c)

$$B_0 = \frac{100\%}{1 + 1.8\% \times \frac{6}{12}} + RV_0.$$

Por seu turno,

$$RV_{6M} = \begin{cases} \frac{S_{6M} - 2,760}{2,760} \Leftarrow S_{6M} > 2,700 \\ 0\% \Leftarrow ELSE \end{cases}$$

$$\Leftrightarrow RV_{6M} = \frac{1}{2,760} \times \begin{cases} S_{6M} - 2,760 \Leftarrow S_{6M} > 2,700 \\ 0 \Leftarrow ELSE \end{cases}$$

$$\Leftrightarrow RV_{6M} = \frac{1}{2,760} \times \begin{cases} S_{6M} - 2,700 - 60 \Leftarrow S_{6M} > 2,700 \\ 0 \Leftarrow ELSE \end{cases}$$

$$\Leftrightarrow RV_{6M} = \frac{1}{2,760} \times \left( \begin{cases} S_{6M} - 2,700 \Leftarrow S_{6M} > 2,700 \\ 0 \Leftarrow ELSE \end{cases} - \begin{cases} 60 \Leftarrow S_{6M} > 2,700 \\ 0 \Leftarrow ELSE \end{cases} \right)$$

$$\Leftrightarrow RV_{6M} = \frac{1}{2,760} \times (c_{6M}(S_{6M}; 2,700; 6M) - D_{6M}(1)(S_{6M}; 2,700; 6M; M = 60)).$$

Portanto,

$$RV_0 = \frac{1}{2,760} \times (c_0(S_0; 2,700; 6M) - D_0(1)(S_0; 2,700; 6M; M = 60)).$$

Utilizando o enunciado,

$$c_0(S_0; 2,700; 6M) = 220.11;$$

Relativamente à cash-or-nothing call,

$$D_0(1)(S_0; 2,700; 6M; M = 60) = 60 \times e^{-1.792\% \times 0.5} \times \Phi[d_2^M(2,700)],$$

sendo

$$\begin{aligned} \Phi[d_2^M(2,700)] &= \Phi \left[ \frac{\ln\left(\frac{2,760}{2,700}\right) + \left(1.792\% - 2\% - \frac{(0.25)^2}{2}\right) \times 0.5}{0.2 \times \sqrt{0.5}} \right] \\ &= \Phi(0.0301) \\ &\cong \Phi(0.03) \\ &= 0.5120. \end{aligned}$$

$$D_0(1)(S_0; 2,700; 6M; M = 60) = 60 \times e^{-1.792\% \times 0.5} \times 0.5120 \cong 30.4454.$$

Em suma,



$$RV_0 = \frac{1}{2,760} \times (220.11 - 30.4454) \cong 6.87\%,$$

e

$$B_0 = 99.11\% + 6.87\% \cong 105.98\% > 100\% \Rightarrow \text{Investir.}$$

d)

$$B_0 = \frac{100\%}{1 + 1.8\% \times \frac{6}{12}} + RV_0.$$

Relativamente à componente de remuneração variável,

$$RV_{6M} = \begin{cases} 40\% \times \max\left(0\%; \frac{S_{6M} - S_0}{S_0}\right) \Leftarrow \inf_{0 < u \leq 0.5} (S_u) > 2,700 \\ 0\% \Leftarrow \inf_{0 < u \leq 0.5} (S_u) \leq 2,700 \end{cases}$$

$$= \frac{40\%}{S_0} \times \begin{cases} \max(0; S_{6M} - S_0) \Leftarrow \inf_{0 < u \leq 0.5} (S_u) > 2,700 \\ 0 \Leftarrow \inf_{0 < u \leq 0.5} (S_u) \leq 2,700 \end{cases}$$

$$= \frac{40\%}{S_0} \times DO(1)_{6M}(S_{6M}; X = S_0; H = 2,700; R = 0; T = 0.5y).$$

Portanto, a remuneração variável é dada por uma down-and-out zero rebate call, i.e.

$$RV_0 = \frac{40\%}{S_0} \times DO(1)_0(S_0; X = 2,760; H = 2,700; R = 0; T = 0.5y).$$

A down-and-in call sem rebate é avaliada com base na seguinte expressão:

$$DO(1)_t(S; X; H; T)$$

$$= c_t(S_t; \max(H; X); T) - \left(\frac{H}{S_t}\right)^{\frac{2\mu}{\sigma^2}} c_t\left(\frac{H^2}{S_t}; \max(H; X); T\right) \\ + [\max(H; X) - X] e^{-r\tau} \left\{ \Phi[d_2^M(S_t, H)] - \left(\frac{H}{S_t}\right)^{\frac{2\mu}{\sigma^2}} \Phi[d_2^M(H, S_t)] \right\}$$

Atendendo a que  $X = 2,760 > H = 2,700$ , então a expressão anterior pode ainda ser simplificada:

$$c_t^{do}(S; X; H < X; T) = c_t(S_t; X; T) - \left(\frac{H}{S_t}\right)^{\frac{2\mu}{\sigma^2}} c_t\left(\frac{H^2}{S_t}; X; T\right).$$

Considerando os dados em apreço,

$$c_0^{do}(S_0; X = 2,760; H = 2,700; R = 0; T = 0.5y)$$

$$= c_0(S_0 = 2,760; X = 2,760; T = 0.5y) - \left(\frac{2,700}{2,760}\right)^{\frac{2\mu}{\sigma^2}} c_t\left(\frac{2,700^2}{2,760}; X = 2,760; T = 0.5y\right).$$

Visto que o valor de uma standard call é uma função homogénea de grau 1 no spot e no strike, então

$$c_0^{do}(S_0; X = 2,760; H = 2,700; R = 0; T = 0.5y)$$

$$= c_0(S_0 = 2,760; X = 2,760; T = 0.5y)$$

$$- \left(\frac{2,700}{2,760}\right)^{\frac{2\mu}{\sigma^2}} \times \frac{(2,700)^2}{(2,760)^2} c_0\left(2,760; 2,760 \times \frac{(2,760)^2}{(2,700)^2} \cong 2,884.03; T = 0.5y\right).$$

Visto que

$$\mu = 1.792\% - 2\% - \frac{(0.25)^2}{2} \cong -0.03333,$$

e utilizando os dados do enunciado, então

$$\begin{aligned}
& c_0^{do}(S_0; X = 2,760; H = 2,700; R = 0; T = 0.5y) \\
&= 191.14 - \left( \frac{2,700}{2,760} \right)^{\frac{2 \times (-0.03333)}{(0.25)^2}} \times \frac{(2,700)^2}{(2,760)^2} \times 140.33 \\
&\cong 53.659208.
\end{aligned}$$

Em suma,

$$RV_0 = \frac{40\%}{2,760} \times 53.659208 \cong 0.78\%,$$

e

$$B_0 = 99.11\% + 0.78\% \cong 99.89\% < 100\% \Rightarrow \text{Não investir.}$$

e)

$$B_0 = \frac{100\%}{1 + 1.8\% \times \frac{6}{12}} + RV_0.$$

$$RV_{6M} = 10\% \times 1_{\left\{ \sup_{0 < u \leq 0.5} (S_u) \geq 2,800 \right\}}.$$

Portanto,

$$\begin{aligned}
RV_0 &= 10\% \times e^{-1.792\% \times 0.5} \times Q \left[ \sup_{0 < u \leq 0.5} (S_u) \geq 2,800 \right] \\
&= 10\% \times e^{-1.792\% \times 0.5} \times Q \left[ \sup_{0 < u \leq 0.5} \left( \ln \left( \frac{S_u}{S_0} \right) \right) \geq \ln \left( \frac{2,800}{S_0} \right) \right] \\
&= 10\% \times e^{-1.792\% \times 0.5} \times \left\{ 1 - Q \left[ \sup_{0 < u \leq 0.5} \left( \ln \left( \frac{S_u}{S_0} \right) \right) < \ln \left( \frac{2,800}{S_0} \right) \right] \right\}.
\end{aligned}$$

Via Proposição 45,

$$\begin{aligned}
& Q \left[ \sup_{0 < u \leq 0.5} \left( \ln \left( \frac{S_u}{S_0} \right) \right) < \ln \left( \frac{2,800}{S_0} \right) \right] \\
&= \Phi \left[ -d_2^M(2,760; 2,800) \right] - \left( \frac{2,800}{2,760} \right)^{\frac{2 \times (-0.03333)}{(0.25)^2}} \Phi \left[ d_2^M(2,800; 2,760) \right] \\
&= \Phi \left[ \frac{\ln \left( 2,760 / 2,800 \right) + \left( 1.792\% - 2\% - \frac{(0.25)^2}{2} \right) \times 0.5}{0.25 \times \sqrt{0.5}} \right] \\
&\quad - \left( \frac{2,800}{2,760} \right)^{\frac{2 \times (-0.03333)}{(0.25)^2}} \Phi \left[ \frac{\ln \left( 2,800 / 2,760 \right) + \left( 1.792\% - 2\% - \frac{(0.25)^2}{2} \right) \times 0.5}{0.25 \times \sqrt{0.5}} \right] \\
&= \Phi(-0.1757) - \left( \frac{2,800}{2,760} \right)^{\frac{2 \times (-0.03333)}{(0.25)^2}} \Phi(-0.0129) \\
&\cong 0.072278173.
\end{aligned}$$

Portanto,

$$RV_0 = 10\% \times e^{-1.792\% \times 0.5} \times \{1 - 0.072278173\}$$

$$\cong 9.194\%.$$

$$B_0 = 99.11\% + 9.194\% = 108.30\% > 100\% \Rightarrow \text{Depositar.}$$

f)

$$B_0 = \frac{100\%}{1 + 1.8\% \times \frac{6}{12}} + RV_0.$$

Daqui a 6 meses:

$$RV_{6M} = 60\% \times \begin{cases} \frac{S_{3M} - S_{6M}}{S_{3M}} \Leftarrow S_{6M} < S_{3M} \\ 0\% \Leftarrow ELSE \end{cases}$$

$$= \frac{60\%}{S_{3M}} \times \begin{cases} S_{3M} - S_{6M} \Leftarrow S_{6M} < S_{3M} \\ 0 \Leftarrow ELSE \end{cases}$$

$$= \frac{60\%}{S_{3M}} \times p_{6M}(S_{6M}; X = S_{3M}; T = 6M)$$

$$RV_{3M} = \frac{60\%}{S_{3M}} \times p_{3M}(S_{3M}; X = S_{3M}; T = 6M) = 60\% \times p_{3M}(1; X = 1; T = 6M)$$

Consequentemente,

$$RV_0 = \frac{60\% \times p_{3M}(1; X = 1; T = 6M)}{1 + 1.4\% \times \frac{3}{12}}.$$

Visto que o valor de uma standard put é uma função homogénea de grau 1 no spot e no strike, então

$$p_{3M}(1; X = 1; T = 6M)$$

$$= \frac{p_0(2,760; X = 2,760; T = 3M)}{2,760}$$

$$= \frac{137.61}{2,760} \cong 4.986\%.$$

Portanto,

$$RV_0 = \frac{60\% \times 4.986\%}{1 + 1.4\% \times \frac{3}{12}} \cong 2.981\%,$$

e

$$B_0 = 99.11\% + 2.981\% = 102.09\%.$$

Margem de intermediação = 103.00% - 102.09% = 0.91%.