OPÇÕES EXÓTICAS
MSc MATEMÁTICA FINANCEIRA 2018/19
EXAME - Resolução

30/07/19 **Duração: 2.5 horas**

<u>CASO 1</u>

a) Prove the valuation formula for a *European-style down-and-in call* with a barrier level (L) higher than the *strike* (X) and without *rebate*.

Utilizando a equação (129) dos apontamentos,

Sabemos que:

$$DI(1)_t(S;X;L>X;R=0;T)$$

$$= e^{-r\tau}\int_{\ln\left(\frac{X}{S_t}\right)}^{\ln\left(\frac{L}{S_t}\right)}\left(S_t e^y - X\right)\phi\left[y;\mu\tau,\sigma\sqrt{\tau}\right]dy$$

$$+ e^{-r\tau}\int_{\ln\left(\frac{L}{S_t}\right)}^{\infty}\left(S_t e^y - X\right)\left(\frac{L}{S_t}\right)^{\frac{2\mu}{\sigma^2}}\phi\left[y;2\ln\left(\frac{L}{S_t}\right)+\mu\tau,\sigma\sqrt{\tau}\right]dy.$$

Fazendo a mudança de variável de integração $z = y - 2\ln\left(\frac{L}{S_t}\right)$,

$DI(1)_t(S;X;L>X;R=0;T)$

$$= e^{-r\tau}\int_{\infty}^{\ln\left(\frac{L}{S_t}\right)}(S_te^y - X)\phi[y;\mu\tau,\sigma\sqrt{\tau}]dy - e^{-r\tau}\int_{\infty}^{\ln\left(\frac{X}{S_t}\right)}(S_te^y - X)\phi[y;\mu\tau,\sigma\sqrt{\tau}]dy$$

$$+\left(\frac{L}{S_t}\right)^{\frac{2\mu}{\sigma^2}} e^{-r\tau}\int_{\ln\left(\frac{L}{L^2/S_t}\right)}^{+\infty}\left[S_t\left(\frac{L}{S_t}\right)^2 e^z - X\right]\phi(z;\mu\tau,\sigma\sqrt{\tau})dz$$

$$= -e^{-r\tau}\int_{\infty}^{\ln\left(\frac{L}{S_t}\right)}[(L - S_te^y) + (X - L)]\phi[y;\mu\tau,\sigma\sqrt{\tau}]dy$$

$$+ e^{-r\tau}\int_{\infty}^{\ln\left(\frac{X}{S_t}\right)}(X - S_te^y)\phi[y;\mu\tau,\sigma\sqrt{\tau}]dy$$

$$+\left(\frac{L}{S_t}\right)^{\frac{2\mu}{\sigma^2}} e^{-r\tau}\int_{\ln\left(\frac{L}{L^2/S_t}\right)}^{+\infty}\left[\left(\frac{L^2}{S_t}e^z - L\right) + (L - X)\right]\phi(z;\mu\tau,\sigma\sqrt{\tau})dz$$

$$= -p_t(S_t,L,T) - e^{-r\tau}(X - L)\Phi\left[\frac{\ln\left(\frac{L}{S_t}\right) - \mu\tau}{\sigma\sqrt{\tau}}\right]$$

$$+ p_t(S_t,X,T)$$

$$+\left(\frac{L}{S_t}\right)^{\frac{2\mu}{\sigma^2}} c_t\left(\frac{L^2}{S_t},L,T\right) + e^{-r\tau}(L - X)\Phi\left[-\frac{\ln\left(\frac{S_t}{L}\right) - \mu\tau}{\sigma\sqrt{\tau}}\right].$$

b) Consider a range-accrual note with bullet redemption (at face value) and a annual coupon equal to 8% times the number of coupon days that the 3-month US-Libor is between 2.5% and 3%, divided by the number of working days in the coupon period. Assume that today the overnight US interest rate is equal to 2% (ACT/365), the range-accrual note will expire tomorrow and the last coupon period contains 260 working days, from which 219 days have had the 3-month US-Libor between 2.5% and 3%. Please formalize the fair value of the range-accrual note.

The fair value of the range-accrual note is simply equal to the present value of the last coupon + the redemption of the fair value:

$$V_0 = \frac{100\% + 8\% \times {}^{219}\!/_{260}}{1 + 2\% \times {}^{1}\!/_{365}}$$

$$+ RD_0\left(L\left(0; {}^{1}\!/_{365}; {}^{1}\!/_{365} + 0.25\right), X_l = 2.5\%, X_u = 3\%, T = {}^{1}\!/_{365}, M = \frac{8\%}{260} \right),$$

where $L\left(0; {}^{1}\!/_{365}; {}^{1}\!/_{365} + 0.25\right)$ is the 3-month forward US Libor to start tomorrow.

c) Admita ter vendido uma *as-you-like-it option* simples (com vencimento no momento T_2) e ter feito o *static hedging* desta posição curta mediante a compra de uma *put standard* com vencimento no momento posterior T_2 e de *calls standard* com vencimento no momento anterior T_1. Admita estar agora no momento T_1 e o comprador da *as-you-like-it option* simples decidir escolher deter uma *call*. Enuncie as operações a desencadear no momento T_1 bem como todos os *cash flows* a elas associados.

c.1) We must sell the standard put, yielding a cash-in flow of

$$p_{T_1}\left(S_{T_1}, X, T_2\right).$$

Given the put-call parity relation,

$$c_{T_1}\left(S_{T_1}, X, T_2\right) - p_{T_1}\left(S_{T_1}, X, T_2\right) = S_{T_1} e^{-q(T_2 - T_1)} - X e^{-r(T_2 - T_1)},$$

and since the owner of the *simple as-you-like-it option* decided to choose a standard call, ie since

$$c_{T_1}\left(S_{T_1}, X, T_2\right) > p_{T_1}\left(S_{T_1}, X, T_2\right),$$

then we may conclude that

$$S_{T_1} e^{-q(T_2 - T_1)} > X e^{-r(T_2 - T_1)},$$

ie

$$X e^{-(r-q)(T_2 - T_1)} < S_{T_1}.$$

c.2) We must exercise the $e^{-q(T_2 - T_1)}$ standard calls, obtaining a payoff equal to

$$e^{-q(T_2 - T_1)} c_{T_1}\left(S_{T_1}, X e^{-(r-q)(T_2 - T_1)}, T_1\right) = e^{-q(T_2 - T_1)} max\left(X e^{-(r-q)(T_2 - T_1)} - S_{T_1}; 0\right)$$

$$= e^{-q(T_2 - T_1)}\left(X e^{-(r-q)(T_2 - T_1)} - S_{T_1}\right) = X e^{-r(T_2 - T_1)} - S_{T_1} e^{-q(T_2 - T_1)},$$

because $X e^{-(r-q)(T_2 - T_1)} < S_{T_1}$.

c.3) These two cash-in flows should be enough to buy the standard call with maturity at time T_2:

$$p_{T_1}\left(S_{T_1}, X, T_2\right) + Xe^{-r(T_2-T_1)} - S_{T_1}e^{-q(T_2-T_1)} = c_{T_1}\left(S_{T_1}, X, T_2\right),$$

where the equality follows from the put-call parity.

CASO 2

a)

$\boldsymbol{r}$: $\boldsymbol{e^{r\times 0.25}} = \mathbf{1} + 2.5\% \times {}^{3}/_{\mathbf{12}} \Rightarrow \boldsymbol{r} = \mathbf{4} \times \boldsymbol{ln}\left(\mathbf{1} + \mathbf{2.5}\% \times {}^{3}/_{\mathbf{12}}\right) \cong 2.4922\%$

.1º valor:

$\boldsymbol{S_{4,10}} = 3{,}005.34 \times \boldsymbol{exp}\left\{\left[\mathbf{2.4922}\% - \mathbf{2}\% - \frac{(\mathbf{0.25})^2}{\mathbf{2}}\right] \times \frac{\mathbf{0.25}}{\mathbf{6}} + \mathbf{0.8518} \times \mathbf{0.25} \times \sqrt{\frac{\mathbf{0.25}}{\mathbf{6}}}\right\}$

$\Leftrightarrow \boldsymbol{S_{4,10}} \cong 3{,}135.42.$

.2º valor:

$$V_{6,1} = max(2{,}940 - \ 2{,}334.25\ ) \times 1_{\{Smax<3{,}000\}}$$
$$= \ 605.75\ \times 1_{\{\ 2{,}569.84\ <3{,}000\}}$$
$$\cong \ 605.75\ .$$

.3º valor:

$$V_{6,6} = max(2{,}940 - \ 2{,}345.53\ ) \times 1_{\{Smax<3{,}000\}}$$
$$= \ 594.47\ \times 1_{\{\ 2{,}569.84\ <3{,}000\}}$$
$$\cong \ 594.47\ .$$

.4º valor:

$$V_{6,7} = max(2{,}940 - \ 2{,}759.64\ ) \times 1_{\{\ 3{,}281.49\ <3{,}000\}}$$
$$\cong 0.$$

b)

$$\hat{V}_0 = e^{-2.4922\% \times 0.25} \times \frac{1{,}848.65}{10} \cong 183.72\,.$$

$$\sigma(\hat{V}_0) = \frac{e^{-2.4922\% \times 0.25}}{\sqrt{10}} \times \sqrt{\frac{889{,}941.49 - {(1{,}848.65)^2}/{10}}{10-1}} \cong 77.56\,.$$

c)

Existem apenas duas simulações (#1 e #6) em que há o evento de knock-out via barreira inferior (na medida em que a cotação mínima é menor ou igual à barreira inferior definida, ie. 2,350).

Nestas duas simulações o payoff obter pelo detentor da opção será não o definido na al~inea a) mas sim de zero para cada simulação.

Assim sendo, o valor actual da opção passará a ser igual a

$$\begin{aligned}\hat{V}_0 &= e^{-2.4922\% \times 0.25} \times \frac{1{,}848.65\text{- }605.75\text{ - }594.47}{10} \\ &= e^{-2.4922\% \times 0.25} \times \frac{648.43}{10} \\ &\cong 64.44\,.\end{aligned}$$

<u>CASO 3</u>

a) Formulate a trading decision for a (USD denominated) bank deposit with expiry date after 6 months, and with a variable return equal to 2% if the S&P500 index decreases (after 6 months) below 2,900 index points.

$$r: \quad e^{r \times 6/12} = 1 + 2.6\% \times 6/12 \Rightarrow r = 2 \times \ln\left(1 + 2.6\%/2\right) \cong 2.5832\%.$$

Concerning the variable return component, and denoting the S&P500 index price by "S", then

$$VR_{6M} = \begin{cases} 2\% \Leftarrow S_{6M} < 2{,}900 \\ 0\% \Leftarrow ELSE \end{cases}$$

$$= 2\% \times p^d_{6M}\left(S; X = 2{,}900; T = 6M; M = 1\right).$$

Therefore,

$$B_0 = 100\% \times e^{-2.5832\% \times 0.5} + VR_0,$$

and

$$VR_0 = 2\% \times p_0^d \left(S; X = 2{,}900; T = 6M; M = 1\right)$$

$$= 2\% \times e^{-2.5832\% \times 0.5} \times N\left[-d_2^M \left(2{,}940;2{,}900\right)\right]$$

$$N\left[-d_2^M \left(2{,}940;2{,}900\right)\right] = N\left[-\frac{\ln\left(2{,}940 / 2{,}900\right) + \left(2.5832\% - 2\% - (0.25)^2 / 2\right) \times 0.5}{0.2 \times \sqrt{0.5}}\right]$$

$$\begin{aligned} &= N(-0.0056) \\ &= 1 - N(0.0056) \\ &\cong 1 - N(0.01) \\ &= 1 - 0.5040 \\ &= 0.4960. \end{aligned}$$

Hence,

$$VR_0 = 2\% \times e^{-2.5832\% \times 0.5} \times 0.4960 \cong 0.98\%,$$

and

$$B_0 = 98.72\% + 0.98\% \cong 99.70\% < 100\% \Rightarrow \text{Do not invest.}$$

b) Formulate a trading decision for a (USD denominated) bank deposit with expiry date after 6 months but that will be redeemed earlier with a variable return equal to 1% if the S&P500 index ever decreases below 2,900 index points at any moment during the next 6 months; otherwise the return obtained after 6 months will be equal to 0%.

$$B_0 = 101\% \times E_Q\left(e^{-2.5832\% \times (\tau_H - 0)} \times 1_{\{\tau_H \le 0.5\}} \middle| F_t\right) + 100\% \times e^{-2.5832\% \times 0.5} \times Q\left[\inf_{0<u<6M} (S_u) > 2{,}900 \middle| F_0\right],$$

where $\tau_H := \inf\{v > 0 : S_v = 2{,}900\}$.

Hence, the present value of the deposit corresponds to the value of a non-defferable knock-out rebate equal to 101% plus the value of knock-in rebate equal to 100%:

$$B_0 = KONDR(\eta = -1; R = 101\%) + KIR(\eta = -1; R = 100\%).$$

Since the first part of the deposit value corresponds to the present value of a non-deferrable knock-out rebate equal to 101% and associated to a down barrier of 2,900 index points, we can use Proposition 56 of the handouts (with $\eta = -1$):

$$KONDR(\eta = -1; R = 101\%)$$

$$= 101\% \times \left\{ \left(\frac{2{,}900}{2{,}940}\right)^{\frac{\mu-\psi}{0.25^2}} \times N\left[\frac{\ln\left(2{,}900 / 2{,}940\right) - \psi \times 0.5}{0.25 \times \sqrt{0.5}}\right] \right.$$

$$\left. + \left(\frac{2{,}900}{2{,}940}\right)^{\frac{\mu+\psi}{0.25^2}} \times N\left[\frac{\ln\left(2{,}900 / 2{,}940\right) + \psi \times 0.5}{0.25 \times \sqrt{0.5}}\right] \right\},$$

where

$$\mu = r - q - \sigma^2/2 = 2.5832\% - 2\% - (0.25)^2/2 = \text{-}0.02542,$$

$$\psi = \sqrt{\mu^2 + 2\sigma^2 r} = \sqrt{(\text{-}\,0.02542)^2 + 2 \times (0.25)^2 \times 2.5832\%} = 0.062250367.$$

Auxiliary calculus:

$$\begin{aligned} N\left[\frac{\ln\left(2{,}900 / 2{,}940\right) - \psi \times 0.5}{0.25 \times \sqrt{0.5}}\right] &= N\left[\frac{\ln\left(2{,}900 / 2{,}940\right) - 0.062250367 \times 0.5}{0.25 \times \sqrt{0.5}}\right] \\ &= N(\text{-}\,0.25356) \\ &= 1 - N(0.25356) \\ &\cong 1 - N(0.25) \\ &= 1 - 0.5987 \\ &= 0.4013, \end{aligned}$$

$$\begin{aligned} N\left[\frac{\ln\left(2{,}900 / 2{,}940\right) + \psi \times 0.5}{0.25 \times \sqrt{0.5}}\right] &= N\left[\frac{\ln\left(2{,}900 / 2{,}940\right) + 0.062250367 \times 0.5}{0.25 \times \sqrt{0.5}}\right] \\ &= N(0.098578) \\ &\cong N(0.10) \\ &= 0.5398. \end{aligned}$$

Therefore,

$$KONDR(\eta=-1;R=101\%)$$

$$=101\%\times\left\{\left(\frac{2{,}900}{2{,}940}\right)^{\frac{\text{-0.02542}-0.062250367}{0.25^2}}\times 0.4013+\left(\frac{2{,}900}{2{,}940}\right)^{\frac{\text{-0.02542}+0.062250367}{0.25^2}}\times 0.5398\right\}$$

$$\cong 95.2029\%.$$

The second part of the deposit value corresponds to the present value of a knock-in rebate equal to 100% and associated to a down barrier of 2,900 index points, for which we can use Proposition 56 of the handouts (with $\eta=-1$):

$$KIR(\eta=-1;R=100\%)$$

$$=100\%\times e^{-2.5832\%\times 0.5}\times\left\{N\left[d_2^M(2{,}940;2{,}900)\right]-\left(\frac{2{,}900}{2{,}940}\right)^{\frac{2\times(\text{-0.02542})}{0.25^2}}N\left[d_2^M(2{,}900;2{,}940)\right]\right\}.$$

Using the results from the answer to the previous question, then

$$N\left[d_2^M(2{,}940;2{,}900)\right]\cong 1-0.4960=0.5040.$$

$$N\left[d_2^M(2{,}900;2{,}940)\right]=N\left[\frac{\ln\left(2{,}900\big/2{,}940\right)+\left(2.5832\%-2\%-\left(0.25\right)^2\big/2\right)\times 0.5}{0.2\times\sqrt{0.5}}\right]$$

$$\begin{aligned}&=N(\text{-}\,0.1494)\\&=1-N(0.1494)\\&\cong 1-N(0.15)\\&=1-0.5596\\&=0.4404.\end{aligned}$$

Hence,

$KIR(\eta = -1; R = 100\%)$

$$= 100\% \times e^{-2.5832\% \times 0.5} \times \left\{ 0.5040 - \left(\frac{2,900}{2,940} \right)^{\frac{2\times(-0.02542)}{0.25^2}} \times 0.4404 \right\}$$

$\cong 5.5945\%$,

and

$B_0 = 95.2029\% + 5.5945\% \cong 100.80\% > 100\% \Rightarrow$ Invest.

c) Formulate a trading decision for a bond issued at the par value of US\$5,000,000, with bullet redemption (at par value) after 6 months, and with a variable return (to be paid after 6 months) equal to 80% of the devaluation rate of the S&P500 index, if this index never decreases below 2,900 index points during the next 6 months.

$$B_0 = 100\% \times e^{-2.5832\% \times 0.5} + VR_0.$$

Concerning the variable return component,

$$VR_{6M} = \begin{cases} 80\% \times \max\left(0\%; \dfrac{S_0 - S_{6M}}{S_0} \right) \Leftarrow \inf\limits_{0<u\leq 6M} (S_u) > 2,800 \\ 0\% \Leftarrow \inf\limits_{0<u\leq 6M} (S_u) \leq 2,800 \end{cases}$$

$$= \frac{80\%}{S_0} \times \begin{cases} \max(0; S_0 - S_{6M}) \Leftarrow \inf\limits_{0<u\leq 6M} (S_u) > 2,800 \\ 0 \Leftarrow \inf\limits_{0<u\leq 6M} (S_u) \leq 2,800 \end{cases}$$

$$= \frac{80\%}{S_0} \times p_{6M}^{do}(S_{6M}; X = S_0; H = 2,800; R = 0; T = 6M).$$

Therefore, the variable return component is given by a down-and-in zero rebate put, i.e.

$$VR_0 = \frac{80\%}{S_0} \times p_0^{do}(S_{6M}; X = S_0; H = 2,800; R = 0; T = 6M).$$

A down-and-out put without rebate is priced through the following expression:

$$p_t^{d0}(S;X;L;T;R=0)$$

$$=\{p_t(S;X;T)-p_t(S;L;T)-(X-L)e^{-r\tau}N[-d_2^M(S_t,L)]\}1_{\{L<X\}}$$

$$-\left(\frac{L}{S_t}\right)^{\frac{2\mu}{\sigma^2}}\left\{p_t\left(\frac{L^2}{S_t};X;T\right)-p_t\left(\frac{L^2}{S_t};L;T\right)\right.$$

$$-(X-L)e^{-r\tau}N[-d_2^M(L,S_t)]\}1_{\{L<X\}}$$

Since X = 2,940 >L = 2,800, using the data provided, and since $\mu=$ -0.02542 and $\psi=0.062250367$, then:

$$p_0^{do}(S_{6M};X=S_0;H=2{,}800;R=0;T=6M)$$

$$=p_0(2{,}940;2{,}940;6M)-p_0(2{,}940;2{,}900;6M)+(2{,}940-2{,}900)\times e^{-2.5832\%\times 0.5}\times N[-d_2^M(2{,}940;2{,}900)]$$

$$-\left(\frac{2{,}900}{2{,}940}\right)^{\frac{2\times(-0.02542)}{0.25^2}}\left\{p_0\left(\frac{2{,}900^2}{2{,}940};X=2{,}940;T=6M\right)-p_0\left(\frac{2{,}900^2}{2{,}940};L=2{,}900;T=6M\right)\right.$$

$$-(2{,}940-2{,}900)\times e^{-2.5832\%\times 0.5}\times N[-d_2^M(2{,}900;2{,}940)]\}$$

Since the value of a standard call option is an homogeneous function of degree one of the spot and the strike, then

$$p_0^{do}(S_{6M};X=S_0;H=2{,}800;R=0;T=6M)$$

$$=p_0(2{,}940;2{,}940;6M)-p_0(2{,}940;2{,}900;6M)+(2{,}940-2{,}900)\times e^{-2.5832\%\times 0.5}\times N[-d_2^M(2{,}940;2{,}900)]$$

$$-\left(\frac{2{,}900}{2{,}940}\right)^{\frac{2\times(-0.02542)}{0.25^2}}\left\{\frac{2{,}900^2}{2{,}940^2}\times p_0\left(2{,}940;\frac{2{,}940^3}{2{,}900^2}\cong 3{,}021.66;T=6M\right)\right.$$

$$-\frac{2{,}900^2}{2{,}940^2}\times p_0\left(2{,}940;\frac{2{,}940^2}{2{,}900}\cong 2{,}980.55;T=6M\right)$$

$$-(2{,}940-2{,}900)\times e^{-2.5832\%\times 0.5}\times N[-d_2^M(2{,}900;2{,}940)]\}$$

Since

$$N\left[-d_2^M(2{,}940;2{,}900)\right] \cong 0.4960,$$

and

$$N\left[-d_2^M(2{,}900;2{,}940)\right] \cong 1-0.4404 = 0.5596,$$

and using the data from the from the exam sheet, then

$$p_0^{do}\left(S_{6M}; X = S_0; H = 2{,}800; R = 0; T = 6M\right)$$

$$= 200.50 - 180.23 + (2{,}940 - 2{,}900)\times e^{-2.5832\%\times 0.5}\times 0.4960$$

$$-\left(\frac{2{,}900}{2{,}940}\right)^{\frac{2\times(-0.02542)}{0.25^2}}\left\{\frac{2{,}900^2}{2{,}940^2}\times 245{,}60 - \frac{2{,}900^2}{2{,}940^2}\times 222.28\right.$$

$$\left. -(2{,}940 - 2{,}900)\times e^{-2.5832\%\times 0.5}\times 0.5596\right\}$$

$$\cong 0.066474.$$

In summary,

$$VR_0 = \frac{80\%}{2{,}940}\times 0.066474 \cong 0.0018\%,$$

and

$$B_0 = 98.7167\% + 0.0018\% \cong 98.7185\% < 100\% \Rightarrow \text{ Do not invest.}$$

d) Compute the fair value of European-style call on the S&P500 index, with a strike equal to 102% of the index value after 3 months and with a time-to-maturity of 6 month. For this purpose, assume that the FRA 3x6 is now quoted at 2.5%.

We must price the following European-style forward-start call:

$c_0^f\left(S_0 = 2{,}940; X = 1.02 \times S_{3M}; T_2 = 6M\right)$

$= 2{,}940 \times e^{-2\% \times 0.25} \times c_{3M}(1;1.02;6M).$

Concerning the standard call price $c_{3M}(1;1.02;6M)$, we can either use the BSM formula or, more easily, use the market option prices provided.

Since the value of a standard call option is an homogeneous function of degree one of the spot and the strike, and because the the FRA 3x6 is quoted at the current 3-month Libor rate, then

$$\begin{aligned} 2{,}940 \times c_{3M}(1;1.02;6M) &= c_{3M}(2{,}940;1.02 \times 2{,}940;6M) \\ &= c_{3M}(2{,}940; 2{,}998.80;6M) \\ &= 121.35 \end{aligned}$$

Therefore,

$c_0^f\left(S_0 = 2{,}940; X = 1.02 \times S_{3M}; T_2 = 6M\right)$

$= e^{-2\% \times 0.25} \times 121.35$

$= 120{,}75.$