OPÇÕES EXÓTICAS
MSc MATEMÁTICA FINANCEIRA 2015/16
EXAME - Resolução

28/07/16 **Duração: 2.5 horas**

CASO 1

a) Enuncie a formula de avaliação de um contrato que promete pagar, daqui a 1 ano, €10 por cada ponto de índice registado (daqui a 1 ano) pelo índice DAX30, desde que tal índice nunca desça (durante o próximo ano) abaixo de 6000 pontos de índice.

Seja "S" a cotação spot do índice DAX30.

O payoff terminal (i.e. daqui a T=1 ano) deste contrato é igual a:

$$V_T = €10 \times S_T \times 1_{\inf\limits_{0=t<u<T}(S_u)>6000}.$$

Por seu turno, o payoff terminal de uma down-and-out call de estilo europeu e sem rebate é igual a

$$c_T^{do}(S;X;H;T) = (S_T - X)^+ \times 1_{\inf\limits_{t<u<T}(S_u)>H}.$$

Consequentemente,

$$V_T = €10 \times c_T^{do}(S; X = 0; H = 6000; T).$$

Visto que

$$c_t^{do}(S;X;H;T)$$

$$= c_t(S_t; \max(H;X);T) - \left(\frac{H}{S_t}\right)^{\frac{2\mu}{\sigma^2}} c_t\left(\frac{H^2}{S_t}; \max(H;X);T\right)$$

$$+ [\max(H;X) - X]e^{-r\tau}\left\{N\left[d_2^M(S_t, H)\right] - \left(\frac{H}{S_t}\right)^{\frac{2\mu}{\sigma^2}} N\left[d_2^M(H, S_t)\right]\right\},$$

então o valor actual do contrato é igual a:

$$V_t = €10 \times c_t\,(S_t;6000;T) - €10 \times \left(\frac{6000}{S_t}\right)^{\frac{2\mu}{\sigma^2}} c_t\left(\frac{6000^2}{S_t};6000;T\right)$$

$$+ €10 \times 6000 \times e^{-r\tau} \left\{ N\left[d_2^M\,(S_t,6000)\right] - \left(\frac{6000}{S_t}\right)^{\frac{2\mu}{\sigma^2}} N\left[d_2^M\,(6000,S_t)\right] \right\}.$$

b) Considere um depósito a 1 ano que paga semestralmente 80% da taxa de valorização semestral do índice Eurostock50, desde que o índice nunca atinga mais de 120% do valor registado no início de cada semestre. Identifique a fórmula de avaliação da remuneração a liquidar daqui a 1 ano.

Designando por "S" a cotação spot do índice Eurostock50, a remuneração a liquidar daqui a 1 ano é igual a:

$$RV_1(1) = \begin{cases} 80\% \times \max\left(0\%;\frac{S_1 - S_{0.5}}{S_{0.5}}\right) \Leftarrow \sup\limits_{0.5y<u\leq 1y} (S_u) < 120\% \times S_{0.5} \\ 0\% \Leftarrow \sup\limits_{0.5y<u\leq 1y} (S_u) \geq 120\% \times S_{0.51} \end{cases}$$

$$= \frac{80\%}{S_{0.5}} \times \begin{cases} \max(0;S_1 - S_{0.5}) \Leftarrow \sup\limits_{0.5y<u\leq 1y} < 1.2 S_{0.5} \\ 0 \Leftarrow \sup\limits_{0.5y<u\leq 1y} \geq 1.2 S_{0.5} \end{cases}$$

$$= \frac{80\%}{S_{0.5}} \times c_1^{uo}\left(S_1;X = S_{0.5};H = 1.2 S_{0.5};R = 0;T = 1y\right).$$

Portanto, a remuneração variável envolve uma forward start up-and-out call option.

Ao invés de tentarmos calcular directamente o valor actual da remuneração variável, é mais fácil avaliar a remuneração variável daqui a 6 meses, pois nessa data o valor do strike e da barreira já serão conhecidos.

Daqui a 6 meses, o valor da remuneração variável a liquidar daqui a 1 ano é igual a:

$$
\begin{aligned}
RV_1(0.5) &= P(0.5,1)\times E_Q\left[\frac{80\%}{S_{0.5}}\times c_1^{uo}(S_1; X = S_{0.5}; H = 1.2S_{0.5}; R = 0; T = 1y)\middle| F_{0.5}\right] \\
&= \frac{80\%}{S_{0.5}}\times P(0.5,1)\times E_Q\left[c_1^{uo}(S_1; X = S_{0.5}; H = 1.2S_{0.5}; R = 0; T = 1y)F_{0.5}\right] \\
&= \frac{80\%}{S_{0.5}}\times c_{0.5}^{uo}(S_{0.5}; X = S_{0.5}; H = 1.2S_{0.5}; R = 0; T = 1y) \\
&= \frac{80\%}{S_{0.5}}\times S_{0.5}\times c_{0.5}^{uo}(S = 1; X = 1; H = 1.2; R = 0; T = 1y) \\
&= 80\%\times c_{0.5}^{uo}(S = 1; X = 1; H = 1.2; R = 0; T = 1y),
\end{aligned}
$$

onde $P(0.5,1)$ é o factor de desconto a 6 meses em vigor daqui a 6 meses, a segunda igualdade resulta do factor de $S_{0.5}$ já ser conhecido daqui a 6 meses, e a última linha é justificada pois uma up-and-out call é uma função homogénea de grau 1 no spot, strike e barreira. Este facto pode ser comprovado com base na formula de avaliação da up-and-out call (sem rebate):

$$c_t^{uo}(S; X; H > X; R = 0, T)$$

$$= c_t(S; X; T) - c_t(S; H; T) - (H - X)e^{-r\tau}N\left[d_2^M(S_t, H)\right]$$

$$-\left(\frac{H}{S_t}\right)^{\frac{2\mu}{\sigma^2}}\left\{c_t\left(\frac{H^2}{S_t}; X; T\right) - c_t\left(\frac{H^2}{S_t}; H; T\right) - (H - X)e^{-r\tau}N\left[d_2^M(H, S_t)\right]\right\}.$$

Para qualquer $\lambda \in \Re_+$, então

$$c_t^{uo}(\lambda S; \lambda X; \lambda H > \lambda X; R = 0, T)$$

$$= c_t(\lambda S; \lambda X; T) - c_t(\lambda S; \lambda H; T) - (\lambda H - \lambda X)e^{-r\tau}N\left[d_2^M(\lambda S_t, \lambda H)\right]$$

$$-\left(\frac{\lambda H}{\lambda S_t}\right)^{\frac{2\mu}{\sigma^2}}\left\{c_t\left(\frac{\lambda^2 H^2}{\lambda S_t}; \lambda X; T\right) - c_t\left(\frac{\lambda^2 H^2}{\lambda S_t}; \lambda H; T\right) - (\lambda H - \lambda X)e^{-r\tau}N\left[d_2^M(\lambda H, \lambda S_t)\right]\right\}$$

$$= c_t^{uo}(S; X; H > X; R = 0, T).$$

c) Deduza a fórmula de avaliação de uma *European-style up-and-in put* com uma barreira (U) superior ao *strike* (X) e sem *rebate*.

Sabemos que:

$$UI(-1)_t(S;X;U>X;R=0;T)$$

$$=e^{-r\tau}\int_{-\infty}^{\ln\left(\frac{X}{S_t}\right)}\left(X-S_te^y\right)\left(\frac{U}{S_t}\right)^{\frac{2\mu}{\sigma^2}}\phi\left[y;2\ln\left(\frac{U}{S_t}\right)+\mu\tau,\sigma\sqrt{\tau}\right]dy.$$

Fazendo a mudança de variável de integração $z=y-2\ln\left(\frac{U}{S_t}\right)$,

$$UI(-1)_t(S;X;U>X;R=0;T)$$

$$=\left(\frac{U}{S_t}\right)^{\frac{2\mu}{\sigma^2}}e^{-r\tau}\int_{-\infty}^{\ln\left(\frac{X}{U^2/S_t}\right)}\left[X-S_t\left(\frac{U}{S_t}\right)^2e^z\right]\phi\left(z;\mu\tau,\sigma\sqrt{\tau}\right)dz$$

$$=\left(\frac{U}{S_t}\right)^{\frac{2\mu}{\sigma^2}}e^{-r\tau}\int_{-\infty}^{\ln\left(\frac{X}{U^2/S_t}\right)}\left[X-\frac{U^2}{S_t}e^z\right]\phi\left(z;\mu\tau,\sigma\sqrt{\tau}\right)dz$$

$$=\left(\frac{U}{S_t}\right)^{\frac{2\mu}{\sigma^2}}p_t\left(\frac{U^2}{S_t},X,T\right).$$

<u>CASO 2</u>

a)

$r:\quad e^{r\times 6/12}=1-0.2\%\times 6/12\Rightarrow r=\frac{12}{6}\ln\left(1-0.2\%\times 6/12\right)\cong -0.2\%.$

$$S_{5,5} = 8{,}818.29 \times \exp\left\{\left[-0.2\% - 1\% - \frac{(0.3)^2}{2}\right] \times \frac{0.5}{6} + (-0.5496) \times 0.3 \times \sqrt{\frac{0.5}{6}}\right\}$$

$$S_{3,9} \cong 8{,}368.55.$$

$$V_{6,6} = \max(10{,}757.81 - 9{,}630;0) \times 1_{\{S\min > 9{,}500\}}$$

$$= 1{,}127.81 \times 1_{\{9{,}754.87 > 9{,}500\}} = 1{,}127.81.$$

$$V_{6,9} = \max(9{,}965.23 - 9{,}630;0) \times 1_{\{S\min > 9{,}500\}}$$

$$= 335.23 \times 1_{\{9{,}926.82 > 9{,}500\}} = 335.23.$$

b)

$$\hat{V}_0 = e^{0.2\% \times 0.5} \times \frac{3{,}084.84}{10} \cong 308.79.$$

$$\sigma(\hat{V}_0) = \frac{e^{0.2\% \times 0.5}}{\sqrt{10}} \times \sqrt{\frac{4{,}014{,}545.99 - (3{,}084.84)^2 / 10}{10 - 1}} \cong 184.66.$$

c)

Valor actual do depósito bancário:

$$B_0 = \frac{99\%}{1 - 0.2\% \times 6/12} + RV_0.$$

Por seu turno,

$$RV_{6,j} = 50\% \times \max\left(0; \frac{S_{6,j} - 9{,}630}{9{,}630}\right) \times 1_{\{S\min > 9{,}500\}}$$

$$= \frac{50\%}{9{,}630} \times \max(0; S_{6,j} - 9{,}630) \times 1_{\{S\min > 9{,}500\}}$$

$$= \frac{50\%}{9{,}630} \times c_{6,j}^{do}(S_{6,j}, X = 9{,}630, L = 9{,}500, R = 0, T = 6M).$$

Trata-se, portanto, de uma European down-and-out ATM call sobre o índice Dax30, com barreira igual a 9,500 pontos de índice e com vencimento a 6 meses, a qual já foi avaliada na alína b).

Consequentemente,

$$RV_0 = \frac{50\%}{9{,}630} \times c_0^{do}\left(S_0, X = 9{,}630, L = 9{,}500, R = 0, T = 6M\right)$$

$$= \frac{50\%}{9{,}630} \times 308.79$$

$$\cong 1.603\%.$$

Em suma,

$B_0 = 99.099\% + 1.603\%\% = 100.702\% > 100\% \Rightarrow$ Comprar.

CASO 3

a) Formule uma decisão de *trading* para um depósito bancário (denominado em USD) com vencimento a 6 meses e com uma remuneração igual 1% caso o índice S&P500 desça abaixo dos 1,800 pontos de índice em qualquer momento durante os próximos 6 meses.

$$r: \quad e^{r\times 6/12} = 1 + 0.75\% \times 6/12 \Rightarrow r = 2\times \ln\left(1 + 0.75\%/2\right) \cong 0.7486\%.$$

$$B_0 = 100\% \times e^{-0.7486\%\times 0.5} + VR_0,$$

Concerning the variable return component, and denoting the S&P500 index price by "S", then

$$VR_{6M} = \begin{cases} 1\% \Leftarrow \inf\limits_{0<u<6M} (S_u) \leq 1{,}800 \\ 0\% \Leftarrow ELSE \end{cases}$$

$$= 1\% \times 1_{\left\{\inf\limits_{0<u<6M} (S_u) \leq 1{,}800\right\}}$$

Hence,

$$VR_0 = 1\% \times e^{-0.7486\%\times 0.5} \times Q\left[\inf_{0<u<6M} (S_u) \leq 1{,}800\right]$$

Since the variable return value corresponds to the present value of a deferrable knock-out rebate equal to 1% and associated to a down barrier of 1,800 index points, we can use Proposition 54 of the handouts (with $\eta = -1$) to compute the following probability:

$$\Pr\left[\inf_{0<u<6M} (S_u) \leq 1{,}800\right]$$

$$= 1 - \Pr\left[\inf_{0<u<6M} (S_u) > 1{,}800\right]$$

$$= 1 - \left\{ N\left[d_2^M(2{,}000;1{,}800)\right] - \left(\frac{2{,}000}{1{,}800}\right)^{\frac{2\mu}{0.2^2}} N\left[d_2^M(1{,}800;2{,}000)\right] \right\}$$

$$= N\left[-d_2^M(2{,}000;1{,}800)\right] + \left(\frac{2{,}000}{1{,}800}\right)^{\frac{2\mu}{0.2^2}} N\left[d_2^M(1{,}800;2{,}000)\right],$$

where

$$\mu = r - q - \sigma^2/2 = 0.7486\% - 2\% - (0.3)^2/2 = \text{-}0.05751.$$

Auxiliary calculus:

$$N\left[-d_2^M(2{,}000;1{,}800)\right] = N\left[-\frac{\ln\left(2{,}000/1{,}800\right) + \left(0.7486\% - 2\% - (0.3)^2/2\right)\times 0.5}{0.3\times\sqrt{0.5}}\right]$$

$$\begin{aligned}
&= N(\text{-}0.3611)\\
&= 1 - N(0.3611)\\
&\cong 1 - N(0.36)\\
&= 1 - 0.6406\\
&= 0.3594.
\end{aligned}$$

$$N\left[d_2^M(1{,}800;2{,}000)\right] = N\left[\frac{\ln\left(1{,}800/2{,}000\right) + \left(0.7486\% - 2\% - (0.3)^2/2\right)\times 0.5}{0.3\times\sqrt{0.5}}\right]$$

$$\begin{aligned}
&= N(\text{-}0.6322)\\
&\cong N(-0.63)\\
&= 1 - N(0.63)\\
&= 1 - 0.7357\\
&= 0.2643.
\end{aligned}$$

Therefore,

$$\Pr\left[\inf_{0<u<6M}(S_u) \leq 1{,}800\right] = 0.3594 - \left(\frac{2{,}000}{1{,}800}\right)^{\frac{2\times(\text{-}0.05751)}{0.3^2}} \times 0.2643 \cong 0.660624,$$

$$VR_0 = 1\% \times e^{-0.7486\%\times 0.5} \times 0.660624 \cong 0.658\%,$$

and

$$B_0 = 99.63\% + 0.658\% \cong 100.285\% > 100\% \Rightarrow \text{Invest.}$$

b) Formule uma decisão de investimento relativamente a uma obrigação de caixa emitida 1% acima do par com um valor nominal de US$10,000,000, com reembolso *bullet* e ao par daqui a 6 meses e com uma remuneração (a liquidar daqui a 6 meses) igual a 40% da taxa de valorização do índice S&P500, desde que tal índice nunca desça abaixo dos 1,950 pontos durante os próximos 6 meses.

$$B_0 = \frac{100\%}{1 + 0.75\% \times {}^{6}\!/_{12}} + VR_0.$$

Concerning the variable return component,

$$VR_{6M} = \begin{cases} 40\% \times \max\left(0\%; \dfrac{S_{6M} - S_0}{S_0}\right) \Leftarrow \inf\limits_{0<u\le 0.5} \left(S_u\right) > 1{,}950 \\ 0\% \Leftarrow \inf\limits_{0<u\le 0.5} \left(S_u\right) \le 1{,}950 \end{cases}$$

$$= \frac{40\%}{S_0} \times \begin{cases} \max\left(0; S_{6M} - S_0\right) \Leftarrow \inf\limits_{0<u\le 0.5} \left(S_u\right) > 1{,}950 \\ 0 \Leftarrow \inf\limits_{0<u\le 0.5} \left(S_u\right) \le 1{,}950 \end{cases}$$

$$= \frac{40\%}{S_0} \times c_{6M}^{do}\left(S_{6M}; X = S_0; H = 1{,}950; R = 0; T = 0.5y\right).$$

Therefore, the variable return is given by a down-and-out zero rebate call, i.e.

$$VR_0 = \frac{40\%}{S_0} \times c_0^{do}\left(S_0; X = 2{,}000; H = 1{,}950; R = 0; T = 0.5y\right).$$

The down-and-out call without rebate is priced through the following expression:

$$c_t^{do}\left(S; X; H; T\right)$$

$$= c_t\left(S_t; \max(H; X); T\right) - \left(\frac{H}{S_t}\right)^{\frac{2\mu}{\sigma^2}} c_t\left(\frac{H^2}{S_t}; \max(H; X); T\right)$$

$$+ \left[\max(H; X) - X\right] e^{-r\tau} \left\{ N\left[d_2^M\left(S_t, H\right)\right] - \left(\frac{H}{S_t}\right)^{\frac{2\mu}{\sigma^2}} N\left[d_2^M\left(H, S_t\right)\right] \right\}$$

Since X = 2,000 > H = 1,950, then the preview expression can be rewritten as:

$$c_t^{do}\left(S; X; H < X; T\right) = c_t\left(S_t; X; T\right) - \left(\frac{H}{S_t}\right)^{\frac{2\mu}{\sigma^2}} c_t\left(\frac{H^2}{S_t}; X; T\right).$$

Considering the data given, then

$$c_0^{do}(S_0; X = 2{,}000; H = 1{,}950; R = 0; T = 0.5y)$$

$$= c_0(S_0 = 2{,}000; X = 2{,}000; T = 0.5y) - \left(\frac{1{,}950}{2{,}000}\right)^{\frac{2\mu}{\sigma^2}} c_t\left(\frac{1{,}950^2}{2{,}000}; X = 2{,}000; T = 0.5y\right).$$

Since the value of a standard call or put option is an homogeneous function of degree one of the spot and the strike, then

$$c_0^{do}(S_0; X = 2{,}000; H = 1{,}950; R = 0; T = 0.5y)$$

$$= c_0(S_0 = 2{,}000; X = 2{,}000; T = 0.5y)$$

$$-\left(\frac{1{,}950}{2{,}000}\right)^{\frac{2\mu}{\sigma^2}} \times \frac{(1{,}950)^2}{(2{,}000)^2} \times c_0\left(2{,}000; 2{,}000 \times \frac{(2{,}000)^2}{(1{,}950)^2} \cong 2{,}103.88; T = 0.5y\right).$$

Since

$$\mu = r - q - \sigma^2/2 = 0.7486\% - 2\% - (0.3)^2/2 = \text{-}0.05751,$$

And using the data from the from the exam sheet, then

$$c_0^{do}(S_0; X = 2{,}000; H = 1{,}950; R = 0; T = 0.5y)$$

$$= 161.64 - \left(\frac{1{,}950}{2{,}000}\right)^{\frac{2\times(\text{-}0.05751)}{(0.3)^2}} \times \frac{(1{,}950)^2}{(2{,}000)^2} \times 120.32$$

$$\cong 43.497012.$$

In summary,

$$VR_0 = \frac{40\%}{2{,}000} \times 43.497012 \cong 0.87\%,$$

and

$B_0 = 99.63\% + 0.87\% \cong 100.50\% < 101\% \Rightarrow$ Do not invest.

c) Formule uma decisão de *trading* para um depósito bancário (denominado em USD) com vencimento a 6 meses e com uma remuneração igual 1% caso o índice S&P500 desvalorize mais do que 10% daqui a 6 meses.

$$VR_{6M} = \begin{cases} 1\% \Leftarrow S_{6M} < 2{,}000 \times 90\% \\ 0\% \Leftarrow ELSE \end{cases}$$

$$= 1\% \times p_{6M}^{d}\left(S; X = 2{,}000 \times 90\%; T = 6M; M = 1\right).$$

Therefore,

$$B_0 = 100\% \times e^{-0.7486\% \times 0.5} + VR_0,$$

and

$$VR_0 = 1\% \times p_0^{d}\left(S; X = 1{,}800; T = 6M; M = 1\right)$$

$$= 1\% \times e^{-0.7486\% \times 0.5} \times N\left[-d_2^{M}\left(2{,}000; 1{,}800\right)\right].$$

From the answer to question 2-a), we already know that

$$N\left[-d_2^{M}\left(2{,}000; 1{,}800\right)\right] = 0.3594.$$

Hence,

$$VR_0 = 1\% \times e^{-0.7486\% \times 0.5} \times 0.3594 \cong 0.36\%,$$

and

$B_0 = 99.63\% + 0.36\% \cong 99.98\% < 100\% \Rightarrow$ Do not invest.

d) Formule uma estratégia de *static hedging* para o depósito definido na alínea a) capaz de eliminar o risco assumido pelo emitente. Atendendo à estratégia agora definida, considera necessário reformular a avaliação do depósito efectuada na alínea c)?

The static hedging strategy that shall be implemented by the issuer of the deposit is the following:

i. Deposit today 99.63% of the amount given by the client and for a period of 6 month; and

ii. Sell standard puts on the S&P500 index, with a time-to-maturity of 6 months, and a strike of 1,800 index points; and buy standard puts with a higher strike (but as close as possible to the first strike, i.e. of 1,820 index points. In both cases, the number of puts to buy or sell must be equal to 1% of the amount deposited (D) divided by US$20 (i.e. US$1x(1,820-1,800)). In fact, the cash flows associated to the hedging of the VR will be the following:

	Today	After 6 months		
		$S_{6M} < 1{,}800$	$1{,}800 \leq S_{6M} < 1{,}820$	$S_{6M} \geq 1{,}820$
1. VR to pay		1%xD	0	0
2. Hedging				
2.1. Short 1,800 put	$\$82.96 \times \frac{1\% \times D}{\$20}$	$-(1800 - S_{6M}) \times \frac{1\% \times D}{20}$	0	0
2.2. Long 1,820 put	$-\$90.31 \times \frac{1\% \times D}{\$20}$	$(1820 - S_{6M}) \times \frac{1\% \times D}{20}$	$(1820 - S_{6M}) \times \frac{1\% \times D}{20}$	0
Total (2)	$(\$82.96 - \$90.31) \times \frac{1\% \times D}{\$20}$	1%xD	$(1820 - S_{6M}) \times \frac{1\% \times D}{20} > 0$	0

If the issuer implements this static hedging strategy, then the value that the issuer must invest today in the hedging of the VR componente will be equal to (in % of the amount deposited:

$$(\$82.96 - \$90.31) \times \frac{1\%}{\$20} \cong -0.37\%.$$

Consequently,

$B_0 = 99.63\% + 0.37\% \cong 99.99\%$ (instead of 99.98%)