OPÇÕES EXÓTICAS
MSc MATEMÁTICA FINANCEIRA 2017/18
EXAME - Resolução

31/07/18 **Duração: 2.5 horas**

<u>CASO 1</u>

a) Prove the valuation formula for a *European-style down-and-in call* with a barrier level (L) lower than the *strike* (X) and without *rebate*.

Utilizando a equação (129) dos apontamentos,

Sabemos que:

$$DI(1)_t(S;X;L<X;R=0;T)$$

$$=e^{-r\tau}\int_{\ln\left(\frac{X}{S_t}\right)}^{+\infty}\left(S_te^y-X\right)\left(\frac{L}{S_t}\right)^{\frac{2\mu}{\sigma^2}}\phi\left[y;2\ln\left(\frac{L}{S_t}\right)+\mu\tau,\sigma\sqrt{\tau}\right]dy.$$

Fazendo a mudança de variável de integração $z=y-2\ln\left(\frac{L}{S_t}\right)$,

$$DI(1)_t(S;X;L<X;R=0;T)$$

$$=\left(\frac{L}{S_t}\right)^{\frac{2\mu}{\sigma^2}}e^{-r\tau}\int_{\ln\left(\frac{X}{L^2/S_t}\right)}^{+\infty}\left[S_t\left(\frac{L}{S_t}\right)^2e^z-X\right]\phi\left(z;\mu\tau,\sigma\sqrt{\tau}\right)dz$$

$$=\left(\frac{L}{S_t}\right)^{\frac{2\mu}{\sigma^2}}e^{-r\tau}\int_{\ln\left(\frac{X}{L^2/S_t}\right)}^{+\infty}\left[\frac{L^2}{S_t}e^z-X\right]\phi\left(z;\mu\tau,\sigma\sqrt{\tau}\right)dz$$

$$=\left(\frac{L}{S_t}\right)^{\frac{2\mu}{\sigma^2}}c_t\left(\frac{L^2}{S_t},X,T\right).$$

b) Consider a European-style put option on the stock SLB, with maturity after 3 years and strike equal to the stock price after 1 year. Please show that such option contract cannot be more expensive than a plain-vanilla European-style and at-the-money put option on the stock SLB with maturity after 2 years.

The European-style put option on the stock SLB, with maturity after 3 years and strike equal to the stock price after 1 year is a forward start option on the strike whose price today is equal to:

$$p_0^f(S_0, X = S_1, 3y) = S_0 e^{-q\times 1} p_1\ (1,1,3y).$$

Assuming that the interest rate, the dividend yield and the vol are constant over the next 3 years, and using the homogeneity property, then

$$\begin{aligned} p_0^f(S_0, X = S_1, 3y) &= e^{-q\times 1} p_1\ (S_0, S_0, 3y) \\ &= e^{-q\times 1} p_0\ (S_0, S_0, 2y) \\ &\leq p_0\ (S_0, S_0, 2y), \end{aligned}$$

because "q" is nonnegative.

c) Please state the static hedging strategy that shall be adopted to hedge a short position on a *range asset-or-nothing option* using only standard and *cash-or-nothing European-style options*

Combining Propositions 8 and 10 of the handouts,

$$\begin{aligned} RA_t\left(S_t, X_l, X_u, T\right) &= M \cdot S_t \cdot e^{-q\tau} \cdot \left[\Phi\left(d_1^M\left(X_l\right)\right) - \Phi\left(d_1^M\left(X_u\right)\right)\right] \\ &= M \cdot S_t \cdot e^{-q\tau} \cdot \Phi\left(d_1^M\left(X_l\right)\right) - M \cdot S_t \cdot e^{-q\tau} \cdot \Phi\left(d_1^M\left(X_u\right)\right) \\ &= A(1)_t\left(S, X_l, T\right) - A(1)_t\left(S, X_u, T\right). \end{aligned}$$

On the other hand, the BSM formula for a standard call is:

$$\begin{aligned} c_t\left(S, X, T\right) &= S_t \cdot e^{-q\tau} \cdot \Phi\left(d_1^M\left(X\right)\right) - X \cdot e^{-r\tau} \cdot \Phi\left(d_2^M\left(X\right)\right) \\ &= A(1)_t\left(S, X, T\right) - D(1)_t\left(S, X, T; M = X\right), \end{aligned}$$

where the second equalitry follows from Proposition 4 of the handouts.

Consequently,

$$\begin{aligned} RA_t\left(S_t, X_l, X_u, T\right) = {} & c_t\left(S, X_l, T\right) + D(1)_t\left(S, X_l, T; M = X_l\right) \\ & - \left[c_t\left(S, X_u, T\right) + D(1)_t\left(S, X_u, T; M = X_u\right)\right]. \end{aligned}$$

Portanto, para criar uma posição *long artificial range asset-or-nothing option* é necessário:

1) Comprar uma standard call com strike X_l;
2) Comprar uma cash-or-nothing call com contract size igual ao strike X_l;
3) Vender uma standard call com strike X_u;
4) Vender uma cash-or-nothing call com contract size igual ao strike X_u.

<u>CASO 2</u>

a)

$$r: \quad e^{r\times 0.5} = 1 + 1.426\% \times 6/12 \Rightarrow r = 2\times\ln\left(1 + 2.5\% \times 6/12\right) \cong 2.4845\%.$$

.1º valor:

$$2{,}697.85 = 2{,}667.63\times\exp\left\{\left[2.4845\% - 3\% - \frac{(0.2)^2}{2}\right]\times\frac{0.5}{6} + \varepsilon_{2,6}\times 0.2\times\sqrt{\frac{0.5}{6}}\right\}$$

$$\Leftrightarrow \varepsilon_{2,6} = \frac{\ln\left(\frac{2{,}697.85}{2{,}667.63}\right) - \left[2.4845\% - 3\% - \frac{(0.2)^2}{2}\right]\times\frac{0.5}{6}}{0.2\times\sqrt{\frac{0.5}{6}}}$$

$$\Leftrightarrow \varepsilon_{2,6} \cong 0.2314.$$

.2º valor:

$$\begin{aligned} V_{6,2} &= \max(3{,}291.80 - 2{,}800; 0)\times 1_{\{S\min > 2{,}400\}} \\ &= 491.80\times 1_{\{2{,}784.70 > 2{,}400\}} \\ &= 491.80. \end{aligned}$$

.3º valor:

$$
\begin{aligned}
V_{6,6} &= \max(2{,}476.07 - 2{,}800;0)\times 1_{\{S\min > 2{,}400\}}\\
&= 0\times 1_{\{2{,}390.89>2{,}400\}}\\
&= 0\times 0\\
&= 0.
\end{aligned}
$$

b)

$$\hat{V}_0 = e^{-2.4845\%\times 0.5}\times\frac{1{,}212.83}{10}\cong 119.79.$$

$$\sigma\left(\hat{V}_0\right)=\frac{e^{-2.4845\%\times 0.5}}{\sqrt{10}}\times\sqrt{\frac{501{,}806.25-\left(1{,}212.83\right)^2\big/10}{10-1}}\cong 62.$$

c)

Existem apenas duas simulações (#6 e #7) em que há o evento de knock-out (na medida em que a cotação mínima é menor ou igual à barreira definida).

Nestas duas simulações o payoff obter pelo detentor da opção será não de zero (conforme estabelecido na tabela do enunciado) mas sim de 100 pontos de índice para cada simulação. No entanto, e visto que o rebate é non-deferrable, tal payoff ocorrerá não na maturidade mas sim aquando do evento de knock-out: daqui a 3 períodos para a simulação #6 e daqui a 4 períodos para a simulação #7.

Assim sendo, o valor actual da opção passará a ser igual a

$$\hat{V}_0 = \frac{e^{-2.4845\%\times 0.5}\times 1{,}212.83 + e^{-2.4845\%\times 0.5\times\frac{3}{6}}\times 100 + e^{-2.4845\%\times 0.5\times\frac{4}{6}}\times 100}{10}$$

$$= \frac{e^{-2.4845\%\times 0.5}\times 1{,}212.83}{10} + \frac{e^{-2.4845\%\times 0.5\times\frac{3}{6}}\times 100 + e^{-2.4845\%\times 0.5\times\frac{4}{6}}\times 100}{10}$$

19.86; 139.64

$$\cong 119.79 + 198.56$$

$$= 318.34.$$

CASO 3

a) Formulate a trading decision for a (USD denominated) bank deposit with expiry date after 6 months, and with a variable return equal to 2% if the Dow Jones index decreases (after 6 months) below 20,000 index points.

$$r: \quad e^{r\times 6/12} = 1 + 2.5\% \times 6/12 \Rightarrow r = 2 \times \ln\left(1 + 2.5\%/2\right) \cong 2.4845\%.$$

Concerning the variable return component, and denoting the Dow Jones index price by "S", then

$$VR_{6M} = \begin{cases} 2\% \Leftarrow S_{6M} < 20{,}000 \\ 0\% \Leftarrow ELSE \end{cases}$$

$$= 2\% \times p_{6M}^{d}\left(S; X = 20{,}000; T = 6M; M = 1\right).$$

Therefore,

$$B_0 = 100\% \times e^{-2.4845\% \times 0.5} + VR_0,$$

and

$$VR_0 = 2\% \times p_0^{d}\left(S; X = 20{,}000; T = 6M; M = 1\right)$$

$$= 2\% \times e^{-2.4845\% \times 0.5} \times N\left[-d_2^{M}\left(20{,}300; 20{,}000\right)\right].$$

$$N\left[-d_2^{M}\left(20{,}300;20{,}000\right)\right] = N\left[-\frac{\ln\left(20{,}300/20{,}000\right) + \left(2.4845\% - 2\% - (0.2)^2/2\right)\times 0.5}{0.2 \times \sqrt{0.5}}\right]$$

$$\begin{aligned} &= N(-0.0517) \\ &= 1 - N(0.0517) \\ &\cong 1 - N(0.05) \\ &= 1 - 0.5199 \\ &= 0.4801. \end{aligned}$$

Hence,

$$VR_0 = 2\% \times e^{-2.4845\% \times 0.5} \times 0.3594 \cong 0.95\%,$$

and

$B_0 = 98.77\% + 0.95\% \cong 99.71\% < 100\% \Rightarrow$ Do not invest.

b) Formulate a trading decision for a (USD denominated) bank deposit with expiry date after 6 months, and with a variable return equal to 2% (after 6 months) if the Dow Jones index ever decreases below 20,000 index points at any moment during the next 6 months.

$$B_0 = 100\% \times e^{-2.4845\% \times 0.5} + VR_0,$$

$$VR_{6M} = \begin{cases} 2\% \Leftarrow \inf\limits_{0<u<6M} (S_u) < 20,000 \\ 0\% \Leftarrow ELSE \end{cases}$$

$$= 2\% \times 1_{\left\{ \inf\limits_{0<u<6M} (S_u) < 20,000 \right\}}$$

Hence,

$$VR_0 = 2\% \times e^{-2.4845\% \times 0.5} \times Q\left[\inf_{0<u<6M} (S_u) < 20,00 \middle| F_0 \right].$$

Since the variable return value corresponds to the present value of a non-deferrable knock-out rebate equal to 2% and associated to a down barrier of 20,000 index points, we can use Proposition 54 of the handouts (with $\eta = -1$) to compute the following probability:

$$Q\left[\inf_{0<u<6M} (S_u) < 20,00 \middle| F_0 \right]$$

$$= N\left[-d_2^M (20,300; 20,000) \right] + \left(\frac{20,000}{20,300} \right)^{\frac{2\mu}{0.2^2}} N\left[d_2^M (20,000; 20,300) \right],$$

where

$$\mu = r - q - \sigma^2 / 2 = 2.4845\% - 2\% - (0.5)^2 / 2 = -0.01515.$$

Using the results from the answer to the previous question, then

$$N\left[-d_2^M (20,300; 20,000) \right] \cong 0.4801.$$

Auxiliary calculus:

$$N\left[d_2^M(20{,}000;20{,}300)\right] = N\left[\frac{\ln\left(20{,}000 \big/ 20{,}300\right) + \left(2.4845\% - 2\% - (0.2)^2 \big/ 2\right) \times 0.5}{0.2 \times \sqrt{0.5}}\right]$$

$$\begin{aligned}
&= N(-0.1589) \\
&= 1 - N(0.1589) \\
&\cong 1 - N(0.16) \\
&= 1 - 0.5636 \\
&= 0.4364.
\end{aligned}$$

Therefore,

$$Q\left[\inf_{0<u<6M}(S_u) < 20{,}00 \middle| F_0\right] = 0.4801 + \left(\frac{20{,}000}{20{,}300}\right)^{\frac{2\times(-0.01515)}{0.2^2}} \times 0.4364 \cong 0.921232,$$

$$VR_0 = 2\% \times e^{-2.4845\% \times 0.5} \times 0.921232 \cong 1.82\%,$$

and

$$B_0 = 98.77\% + 1.82\% \cong 100.59\% > 100\% \Rightarrow \text{Invest.}$$

c) Compute the fair value of a (USD denominated) bond with a time-to-maturity of one year, bullet redemption (at par value) and a semi-annual coupon equal to 60% of the semi-annual devaluation rate (if any) of the Dow Jones index.

The fair value equals the present value of the principal redemption plus the present value of the two coupons (to be received after 6 and 12 months):

$$B_0 = \frac{100\%}{1 + 2.5\%} + C_{6M}(0) + C_{12M}(0).$$

The first coupon (to be received after 6 months) will be equal to:

$$
\begin{aligned}
C_{6M}(6M) &= 60\% \times \max\left(0\%;\frac{S_0 - S_{6M}}{S_0}\right) \\
&= \frac{60\%}{S_0} \times \max(0\%; S_0 - S_{6M}) \\
&= \frac{60\%}{S_0} \times p_{6M}(S_{6M}; S_0; 6M).
\end{aligned}
$$

Therefore, the present value of such coupon is equal to:

$$
\begin{aligned}
C_{6M}(0) &= \frac{60\%}{S_0} \times p_0(S_0; S_0; 6M) \\
&= \frac{60\%}{20{,}300} \times 1{,}107.45 \\
&\cong 3.273\%,
\end{aligned}
$$

where the second line follows the table with option prices contained in the exam sheet.

The second coupon (to be received after 12 months) will be equal to:

$$
\begin{aligned}
C_{12M}(12M) &= 60\% \times \max\left(0\%;\frac{S_{6M} - S_{12M}}{S_{6M}}\right) \\
&= 60\% \times \max\left(0\%; 1 - \frac{S_{12M}}{S_{6M}}\right) \\
&= 60\% \times p_{12M}^{rf}(S_{12M}; S_{6M}; 12M).
\end{aligned}
$$

Therefore, the present value of such coupon is equal to:

$$
\begin{aligned}
C_{12M}(0) &= 60\% \times p_0^{rf}(S_0; S_{6M}; 12M) \\
&= 60\% \times e^{-2.4845\% \times 0.5} \times p_{6M}(1;1;12M).
\end{aligned}
$$

Since the 6 and 12 month spot rates are the same, the 6x12 forward rate will also be the same. Therefore, and since the value of a standard put option is an homogeneous function of degree one of the spot and the strike,

$$
\begin{aligned}
C_{12M}(0) &= 60\% \times e^{-2.4845\% \times 0.5} \times p_{6M}(1;1;12M) \\
&= \frac{60\% \times e^{-2.4845\% \times 0.5}}{20{,}300} \times p_{6M}(20{,}300;20{,}300;12M) \\
&= \frac{60\% \times e^{-2.4845\% \times 0.5}}{20{,}300} \times p_{0}(20{,}300;20{,}300;6M) \\
&= \frac{60\% \times e^{-2.4845\% \times 0.5}}{20{,}300} \times 1{,}107.45 \\
&\cong 3.233\%,
\end{aligned}
$$

where the fourth equality follows the table with option prices contained in the exam sheet.

In summary,

$$
B_0 = 97.561\% + 3.273\% + 3.233\%
$$

$$
\cong 104.067\%.
$$

d) Formulate a *static hedging* strategy for a short position on the bond defined in the previous question.

Static hedging strategy:

1. To ensure the redemption of principal at year 1, deposit today (for 1 year and @2.5%) $\text{US\$}10{,}000{,}000 \times \frac{100\%}{1+2.5\%} = \text{US\$}10{,}000{,}000 \times 97.561\% = US\$9{,}756{,}100.$

2. To ensure the payment of the first coupon, buy today European-style and ATM puts with maturity after 6 month and contract size equal to $\text{US\$}10{,}000{,}000 \times \frac{60\%}{20{,}300}$.
3. To ensure the payment of the second coupon, deposit today, and for 6 months, the amount $\text{US\$}10{,}000{,}000 \times 3.233\% = \text{US\$}10{,}000{,}000 \times 60\% \times e^{-2.4845\% \times 0.5} \times p_{6M}(1;1;12M)$

4. After 6 months, the compounded value of such deposit will be equal to $\text{US\$}10{,}000{,}000 \times 60\% \times p_{6M}(1;1;12M) = \frac{\text{US\$}10{,}000{,}000 \times 60\%}{S_{6M}} \times p_{6M}(S_{6M};S_{6M};12M)$

5. Hence, after 6 months, we can just buy European-style and ATM puts with maturity at month 12 and contract size equal to

$\frac{\text{US\$}10{,}000{,}000 \times 60\%}{S_{6M}}$. The payoff of such position at month 12 will then be the desired one:

$$\text{US\$}10{,}000{,}000 \times 60\% \times \max\left(0\%; \frac{S_{6M} - S_{12M}}{S_{6M}}\right).$$

e) Formulate a trading decision for a bond issued at the par value of US$10,000,000, with bullet redemption (at par value) after 6 months, and with a variable return (to be paid after 6 months) equal to 30% of the devaluation rate of the Dow Jones index, if this index ever decreases below 20,000 index points during the next 6 months.

$$B_0 = 100\% \times e^{-2.4845\% \times 0.5} + VR_0.$$

Concerning the variable return component,

$$VR_{6M} = \begin{cases} 30\% \times \max\left(0\%; \frac{S_0 - S_{6M}}{S_0}\right) \Leftarrow \inf\limits_{0<u\leq 6M} (S_u) \leq 20{,}000 \\ 0\% \Leftarrow \inf\limits_{0<u\leq 6M} (S_u) > 20{,}000 \end{cases}$$

$$= \frac{30\%}{S_0} \times \begin{cases} \max(0; S_0 - S_{6M}) \Leftarrow \inf\limits_{0<u\leq 6M} (S_u) \leq 20{,}000 \\ 0 \Leftarrow \inf\limits_{0<u\leq 6M} (S_u) > 20{,}000 \end{cases}$$

$$= \frac{30\%}{S_0} \times p_{6M}^{di}(S_{6M}; X = S_0; H = 20{,}000; R = 0; T = 6M).$$

Therefore, the variable return component is given by a down-and-in zero rebate put, i.e.

$$VR_0 = \frac{30\%}{S_0} \times p_0^{di}(S_0; X = S_0; H = 20{,}000; R = 0; T = 6M).$$

A down-and-in put without rebate is priced through the following expression:

$$p_t^{di}\left(S;X;H;T;R=0\right)$$

$$=\left(\frac{H}{S_t}\right)^{\frac{2\mu}{\sigma^2}}\left\{c_t\left(\frac{H^2}{S_t};X;T\right)-c_t\left(\frac{H^2}{S_t};H;T\right)\right.$$

$$\left.+\left(X-H\right)e^{-r\tau}N\left[d_2^M\left(H,S_t\right)\right]\right\}1_{\{H<X\}}$$

$$+\,p_t\left(S;\min(H;X);T\right)+\left[X-\min(H;X)\right]e^{-r\tau}N\left[-d_2^M\left(S_t,H\right)\right]$$

Since X = 20,300 < H = 20,500, then the previous formula can be rewritten as:

$$p_t^{di}\left(S;X;H;T;R=0\right)$$

$$=\left(\frac{H}{S_t}\right)^{\frac{2\mu}{\sigma^2}}\left\{c_t\left(\frac{H^2}{S_t};X;T\right)-c_t\left(\frac{H^2}{S_t};H;T\right)\right.$$

$$\left.+\left(X-H\right)e^{-r\tau}N\left[d_2^M\left(H,S_t\right)\right]\right\}$$

$$+\,p_t\left(S;\min(H;X);T\right)+\left[X-\min(H;X)\right]e^{-r\tau}N\left[-d_2^M\left(S_t,H\right)\right]$$

Using the data provided, and since μ = -0.01515, then:

$$p_0^{di}\left(S_0=20{,}300;X=S_0;H=20{,}000;R=0;T=6M\right)$$

$$=\left(\frac{20{,}000}{20{,}300}\right)^{\frac{2\times(-0.01515)}{0.2^2}}\left\{c_0\left(\frac{20{,}000^2}{20{,}300};X=20{,}300;T=6M\right)-c_0\left(\frac{20{,}000^2}{20{,}300};20{,}000;6M\right)\right.$$

$$\left.+\left(20{,}300-20{,}000\right)\times e^{-2.4845\%\times 0.5}\times N\left[d_2^M\left(20{,}000;20{,}300\right)\right]\right\}$$

$$+\,p_0\left(20{,}300;20{,}000;6M\right)+\left(20{,}300-20{,}000\right)\times e^{-2.4845\%\times 0.5}\times N\left[-d_2^M\left(20{,}300;20{,}000\right)\right]$$

Since the value of a standard call option is an homogeneous function of degree one of the spot and the strike, then

$$p_0^{di}\left(S_0 = 20{,}300; X = S_0; H = 20{,}000; R = 0; T = 6M\right)$$

$$= \left(\frac{20{,}000}{20{,}300}\right)^{\frac{2\times(-0.01515)}{0.2^2}} \left\{\frac{20{,}000^2}{20{,}300^2}\times c_0\left(20{,}300; 20{,}300\times\frac{20{,}300^2}{20{,}000^2}\cong 20{,}913.57; T = 6M\right)\right.$$

$$-\frac{20{,}000^2}{20{,}300^2}\times c_0\left(20{,}300; 20{,}000\times\frac{20{,}300^2}{20{,}000^2}\cong 20{,}604.50; T = 6M\right)$$

$$\left.+\left(20{,}300-20{,}000\right)\times e^{-2.4845\%\times 0.5}\times N\left[d_2^M\left(20{,}000; 20{,}300\right)\right]\right\}$$

$$+ p_0\left(20{,}300; 20{,}000; 6M\right) + \left(20{,}300-20{,}000\right)\times e^{-2.4845\%\times 0.5}\times N\left[-d_2^M\left(20{,}300; 20{,}000\right)\right].$$

Since

$$N\left[-d_2^M\left(20{,}300;20{,}000\right)\right]\cong 0.4801,$$

and

$$N\left[d_2^M\left(20{,}000;20{,}300\right)\right] = 0.4364,$$

and using the data from the from the exam sheet, then

$$p_0^{di}\left(S_0 = 20{,}300; X = S_0; H = 20{,}000; R = 0; T = 6M\right)$$

$$= \left(\frac{20{,}000}{20{,}300}\right)^{\frac{2\times(-0.01515)}{0.2^2}} \left\{\frac{20{,}000^2}{20{,}300^2}\times 891.38 - \frac{20{,}000^2}{20{,}300^2}\times 1{,}018.43\right.$$

$$\left.+\left(20{,}300-20{,}000\right)\times e^{-2.4845\%\times 0.5}\times 0.4364\right\}$$

$$+959.17+\left(20{,}300-20{,}000\right)\times e^{-2.4845\%\times 0.5}\times 0.4801$$

$$\cong 1{,}107.40.$$

In summary,

$$VR_0 = \frac{30\%}{20{,}300}\times 1{,}107.40 \cong 1.64\%,$$

and

$B_0 = 98.77\% + 1.64\% \cong 100.40\% > 100\% \Rightarrow$ Invest.